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ANÁLISE DE MECANISMOS COM RESTRIÇÕES REDUNDANTES ATRAVÉS DA APLICAÇÃO DA TEORIA DE MATROIDES.

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Esta Tese foi julgada aprovada para a obtenção do Título de "Doutor em Engenharia Mecânica", e aprovada em sua forma final pelo Programa de Pós-Graduação em Engenharia Mecânica.

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To my wife and my parents.

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Quelli che s'innamorano di pratica senza scienza son come il nocchiere, che entra in naviglio senza timone o bussola, che mai ha certezza dove si vada.

Leonardo da Vinci

RESUMO

O estudo de mecanismo é uma das áreas mais importantes no projeto de máquinas e os seus problemas podem ser divididos em dois grupos: análise de mecanismos e síntese de mecanismos. O foco desta tese é a análise da topologia de mecanismos, em termos de graus de liberdade e restrições, através da teoria de helicoides e da teoria de matroides. Na tese é elaborada uma modelagem geral dos graus de liberdade e das restrições de um mecanismo, utilizando a representação por helicoides e a adaptação das leis de Kirchhoff proposta por Davies para cadeias cinemáticas. Baseada nesta modelagem, é desenvolvida uma nova metodologia de análise de mecanismos para a eliminação automática das restrições redundantes. Ao mesmo tempo, a teoria de matroides é utilizada na análise dos mecanismos. A tese introduz novos resultados na teoria de mecanismos. Primeiramente, é analisada a escolha dos conjuntos de atuadores válidos para um mecanismo. Dois novos algoritmos são propostos para a enumeração de todos os possíveis conjuntos válidos de atuadores e a para a escolha ótima de um conjunto válido de atuadores com base nas especificações do mecanismo. Posteriormente, são analisados os possíveis mecanismos auto alinhantes derivados de um mecanismo com restrições redundantes. Dois novos algoritmos são propostos para enumeração de todos os possíveis mecanismos auto alinhantes obtidos retirando as restrições redundantes de um dado mecanismo e para escolha ótima de um mecanismo auto alinhante, com base nas suas especificações. Os algoritmos foram implementados no software Sage e apresentam complexidade polinomial. Exemplos de aplicação são apresentados e os resultados validados frente à literatura. Duas contribuições adicionais são também introduzidas: a definição de um invariante cinemático que relaciona a mobilidade com o número de restrições redundantes de um mecanismos e um contraexemplo para a metodologia de análise das restrições redundantes proposta por Reshetov.

Palavras-chave: Teoria de Helicoides, Mecanismo, Auto alinhamento, Atuação, Teoria de Matroides

ABSTRACT

The study of mechanisms is one of the most important areas on which machine design relies. Research in mechanism can be roughly divided into two main problems: mechanism analysis and mechanism synthesis. This thesis focuses on topology analysis of mechanism, by means of screw theory representation of mechanisms. Freedoms and constraints in mechanisms are thus described applying the Kirchhoff's laws adaptation to multibody systems proposed by Davies. Based on this modelling, overconstraint in mechanisms is analysed in terms of free motions and constraints. Two main contributions are proposed along this work, based on matroid theory and linear algebra modelling. First, the actuation schemes of a mechanism are investigated. Two algorithms are proposed for enumerating all valid actuation schemes of an overconstrained mechanism and for selecting an optimal actuation scheme, based on a set of criteria. Second, the self-aligning mechanisms kinematically equivalent to an overconstraint mechanism are investigated. Two new algorithms for enumerating all self-aligning kinematically equivalent mechanisms to an overconstrained one and for selecting an optimal self-aligning topology, based on a set of criteria, are proposed. All algorithms have been implemented in Sage software and run in polynomial time. Examples of applications are presented, and the results obtained validated with literature cases. Moreover, two further contributions are proposed: the definition of an invariant kinematic chain relating mobility and degree of constraint and a counterexample for the methodology proposed by Reshetov.

Palavras-chave: Screw Theory, Mechanism, Self-aligning, Actuation Scheme, Matroid Theory

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LIST OF ABBREVIATIONS AND ACRONYMS

CAD	Computer Aided Design
ERO	Elemenatry Row Operations on a matrix
doc	Degree of constraint
dof	Degree of freedom
ref	Row echelon form of a matrix
rref	Reduced row echelon form of a matrix

NOMENCLATURE

$[\boldsymbol{\Psi}]$	vector of generalised magnitudes of action screws
$\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]_{\lambda\nu,F}$	network unit motion matrix of a coupling network
$\zeta^i_{\not\supseteq j}$	arbitrary wrench belonging to $W^i_{\not\supseteq j}$
C	gross degree of constraint of a coupling network $C = \sum c_i$
C_N	degree of redundant constraint of a coupling network
E	ground set of matroid $\mathcal{M}$
F	gross degree of freedom of a coupling network $F = \sum f_i$
F_N	net degree of freedom of a coupling network
G_A	action graph
G_C	coupling graph
G_M	motion graph
$N(\boldsymbol{A})$	null space of $\boldsymbol{A}$
$R(\boldsymbol{A})$	range of $\boldsymbol{A}$
$W^i_{\not\supseteq j}$	set of wrenches that can be exerted on the moving platform through the actuation of joint j in leg i
$\boldsymbol{\$}^{\boldsymbol{a}}$	motion screw of coupling i
$\boldsymbol{\$}^{\boldsymbol{m}}_{\boldsymbol{i}}$	motion screw of coupling i
$\bar{\boldsymbol{\Delta}}$	reciprocal screw system
$\boldsymbol{\Delta}$	screw system
$\boldsymbol{\$}_{\boldsymbol{\alpha}}, \boldsymbol{\$}_{\boldsymbol{\beta}}, \boldsymbol{\$}_{\boldsymbol{\gamma}}$	principal screws

$h_\alpha, h_\beta, h_\gamma$	pitch of principal screws
$\boldsymbol{S_0}$	position vector
$\boldsymbol{S}$	direction vector
$\boldsymbol{V}$	vector space
$\hat{\$}_i$	normalised screw of coupling i
$\left[{}^{\boldsymbol{i}}\boldsymbol{R_j}\right]$	rotation matrix
$\left[{}^{\boldsymbol{i}}\boldsymbol{T_j}\right]$	linear transformation of a screw **\$** from coordinates system i into j
λ	dimension of screw system
$[\boldsymbol{\psi}]$	vector of generalised magnitudes of motion screws
$[\boldsymbol{A_l}]_{\lambda\nu,1}$	vector of the $\lambda\nu$ action screw components for all ν circuits
$[\boldsymbol{B_M}]_{\nu,F}$	circuit matrix of motion graph G_M
$[\boldsymbol{B}]_{\nu,g}$	circuit matrix of coupling graph G_C
$[\boldsymbol{Q}]_{k,g}$	cutset matrix of coupling graph G_C
$[\boldsymbol{Q}_A]_{k,C}$	cutset matrix of action graph G_A
$[\boldsymbol{Q}_i]_{C,C}$	diagonal matrix whose diagonal elements correspond to row i of cutset matrix $[\boldsymbol{Q}_A]_{k,C}$
$\left[\boldsymbol{\hat{A}_D}\right]_{\lambda,F}$	unit action matrix of a coupling network
$\left[\boldsymbol{\hat{A}_N}\right]_{\lambda\nu,F}$	network unit action matrix of a coupling network
$\left[\boldsymbol{\hat{M}_D}\right]_{\lambda,F}$	unit motion matrix of a coupling network
$\left[\boldsymbol{\hat{M}_N}\right]_{\lambda\nu,F}$	network unit motion matrix of a coupling network
$[\boldsymbol{i}]$	vector of instantaneous currents flowing in the edges
$[\boldsymbol{v}]$	vector of instantaneous potential differences

	of the edges
$\mathbb{R}$	set of real numbers
$\mathcal{B}$	collection of bases of matroid $\mathcal{M}$
$\mathcal{B}(\mathcal{M}_{A_N})$	collection of bases of the matroid $\mathcal{M}_{A_N}$
$\mathcal{B}(\mathcal{M}_{M_N})$	collection of bases of matroid $\mathcal{M}_{M_N}$
$\mathcal{B}^*(\mathcal{M})$	collection of bases of dual matroid $\mathcal{M}^*$
$\mathcal{B}^*(\mathcal{M}_{M_N})$	collection of bases of dual matroid $\mathcal{M}^*_{M_N}$
$\mathcal{C}(\mathcal{M})$	collection of circuits of a matroid $\mathcal{M}$
$\mathcal{F}$	kinematic invariant of kinematic chain
$\mathcal{I}$	set of all independent subset of a matroid $\mathcal{M}$
$\mathcal{M}$	generic matroid
$\mathcal{M}^*$	dual matroid of matroid $\mathcal{M}$
$\mathcal{M}^*_{M_N}$	dual matroid of $\mathcal{M}_{M_N}$
$\mathcal{M}_{M_N}$	matroid defined over matrix $\left[\boldsymbol{\hat{M}_N}\right]$
$\mathcal{M}^*_{M_A}$	dual matroid of $\mathcal{M}_{M_A}$
$\mathcal{M}_{M_A}$	matroid defined over matrix $\left[\boldsymbol{\hat{M}_A}\right]$
$\mathcal{P}$	power expended by an action screw on a motion screw
$\mathcal{W}$	collection of weights attributed to element of matroid $\mathcal{M}$
ν	number of circuits of a graph
ρ	rank function of matroid $\mathcal{M}$
$\{R, S, T, U, V, W\}$	action screw components in axis-coordinates
$\{r, s, t, u, v, w\}$	motion screw components in ray-coordinates
c_i	gross degree of constraint of a direct coupling i
f_i	gross degree of freedom of a direct coupling i

f_{rx}	angular mobility around x axis - Reshetov methodology
f_{ry}	angular mobility around y axis - Reshetov methodology
f_{rz}	angular mobility around z axis - Reshetov methodology
f_{tx}	linear mobility long x axis - Reshetov methodology
f_{ty}	linear mobility long y axis - Reshetov methodology
f_{tz}	linear mobility long z axis - Reshetov methodology
g	number of joints in a coupling network
h	screw pitch
k	number of independent cutsets of a graph
n	number of links in a coupling network
p_i	number of kinematic pairs of class i
r	rank of a matrix
w_i	weigh of element i of matroid $\mathcal{M}$
C	cylindrical pair
P	prismatic pair
R	revolute pair
S	spherical pair
U	universal pair

CONTENTS

1 INTRODUCTION

The study of mechanisms is one of the most important areas on which machine design relies. Research in mechanism can be roughly divided into two main problems: mechanism analysis and mechanism synthesis. Mechanism analysis focuses on investigating the structure principle of mechanisms, with emphasis in the free motions and constraints, in kinematics and dynamics, in order to determine the main equations that rule motions and forces in mechanism and to provide a theoretical basis for evaluating mechanism performance. The synthesis of mechanisms can be defined as the theory and methods of designing new mechanisms satisfying a set of given specifications. It includes structure synthesis, kinematics and dynamics synthesis.

The structure analysis of mechanism, which focuses on the structure principle of mechanism in terms of freedoms and constraints, constitutes a fundamental theoretical basis for further analysis in kinematics, dynamics and synthesis of mechanism. Moreover structure analysis permits the study of mechanism actuation, in terms of determining the correct number of actuators and selecting a feasible set of actuated joints. Therefore, it is essential to conduct further theoretical research into the structure, type and kinematic characteristics of mechanisms, so as to provide more complete and more general modelling of mechanisms for further developing kinematics and dynamics theory, and to provide a theory to design new mechanisms.

Quite different approaches have been proposed in literature for mechanism modelling focusing on structure analysis. In the theoretical study of mechanisms, rigid body position, motion and constraint have been traditionally described commonly in terms of vectors. This approach suggests the use of matrix transformation, vector product and vector differentiation for studying mechanism kinematics and dynamics problems such as speed and acceleration.

More recently, screw theory has been widely used for modelling a mechanism in terms of motions and constraints. First introduced by Ball(1), screw theory was further developed by Hunt(2) and Phillips(3). Innumerous authors have, since then, contributed to screw theory formulation and extension. More specific introduction to screw theory with literature review is presented in Chapter 2.

Screw theory is a powerful mathematical tool for the analysis of spatial mechanisms. A screw can be used to denote the position and orientation of a spatial vector, thus linear and angular velocities of a rigid body, or force and moment can be represented. Moreover, the transformation between screw-based methods and matrix based methods is straightforward, and powerful linear algebra tools can be employed for mechanism analysis and modelling.

This thesis focuses on structure analysis of mechanism, by means of screw theory representation of mechanisms. Freedoms and constraints in mechanisms are thus described applying the Kirchhoff's laws to multibody systems proposed by Davies(4). Based on this modelling, overconstraint in mechanisms is analysed using the representation of mechanisms in terms of freedoms and constraints. A new approach is introduced based on matroid theory and linear algebra modelling. Based on this approach, new algorithms for actuators selection and redundant constraint elimination are introduced.

1.1 MOBILITY AND ACTUATION IN OVERCONSTRAINED MECHANISMS

One of the most important topics in mechanism analysis is the determination of the degree of freedom (*dof*), *i.e.* the mobility of a mechanism. Free motions and constraints analysis, kinematics and dynamics analysis, number and selection of actuators, all depend upon the correct calculation of the mobility of mechanisms.

Thus the focus of this Thesis in determining the *dof* of a mechanism is how to analyse the multi-loop spatial linkages with local overconstraints and mobilities, and how to relate the local structure with the overall characteristics of the mechanism. Many authors have contributed to the study of mobility analysis, mainly from Germany and the Soviet Union. In the late 19th century, Reuleaux(5) (apud Uicker and Shigley(6)) made a precise definition of mechanism in terms of kinematic pairs. He also introduced an equation relating the number of links and kinematic pairs to the mobility of a mechanism. Based on his research two German scholars, Grübler and Kutzbach, proposed mobility equations for mobility calculation, both for planar and spatial mechanism (6).

Most of mechanisms for practical engineering application of that time were generally planar single-*dof* with a few number of loops. For this class of mechanisms, the Grübler-Kutzbach formulation (7), which became the most well-known equation for mobility analysis from the work of Grübler and Kutzbach, could correctly evaluate the mobility for almost all planar and some spatial mechanisms.

However, this equation formulation fails to analyse many overconstrained spatial mechanisms. In overconstrained mechanisms, *i.e.* mechanisms which present redundant constraints, one or more constraints imposed by the joints concurrently constraint the same mobility in the mechanism.

Many different alternative formulations have been proposed in literature for determining mobility. Gogu(8) presented a critical review of the most important contributions to mobility calculation, reporting limitations and counterexamples. Gogu(9) also introduces a new method based on linear transformations for mobility evaluation.

More recently, the increasing interest in lower-mobility overconstrained parallel mechanisms, for which the Grübler-Kutzbach formulation cannot correctly evaluate mobility, turned mobility calculation a critical issue. Parallel mechanisms are constituted by an *end effector*, which is defined as the output link of the mechanism (10), connected to the *base* by a set of independent kinematic chains (11). Each independent kinematic chain, denoted as *limb*, is constituted by a set of joints and links. Usually, a limb is referred to by the set of couplings of the kinematic chain. The more common couplings are: prismatic, referred as P, revolute, R, universal, U, spherical, S and cylindrical, C. Thus, a limb constituted by a revolute, a cylindrical and a revolute couplings in series is usually referred to as RCR.

Finally, recent works in literature converge toward a unified mobility principle. Based on reciprocal screw theory, the works of Zhao et al.(12), Huang, Li and Ding(13), Dai, Huang and Lipkin(14) and Kong, Gosselin and Richard(15) indicate that the mobility formulation is really unified. A more specific review of mobility formulation is presented in Section 2.2.2.

In parallel, the work of Davies (16, 17) introduced an alternative formulation for calculating mobilities and constraints of mechanisms. Based on the graph representation of a kinematic chain, Davies adapts Kirchhoff laws to multibody system in order to analyse kinematics and statics in mechanisms, and correctly evaluate local degree of freedom and constraint. The Davies formulation has been extensively used along this Thesis, and is further reviewed in the next chapters.

The investigation of the actuation schemes of a mechanism, more precisely the evaluation of the number of actuators necessary to control a mechanism and the correct selection of actuated joints, is a problem strictly related to the structure analysis of mechanisms.

Zhao et al.(12) states that the degree of freedom of the end effector of parallel manipulators, referred as *DOF*, is different from the number of actuators needed to control the mechanism. In fact the configuration degree of freedom, referred as *CDOF*, indicates the number of mobilities owned by the whole mechanism, and it represents the number of actuations required to uniquely control the end effector under any configurations. Zhao proposes a new method for investigating the actuations and motions of a mechanism. In (18) and (12) a method for selection of valid actuation schemes is proposed. In this Thesis a new method, original contribution of this work, is introduced and the results compared with the works of Zhao.

Qin and Dai(19) analyse a $2US + UPS$ asymmetrical parallel mechanism in terms of mobility by means of screw theory. Accordingly, actuation schemes are therefore proposed. Gan et al.(20) present the analysis of a new metamorphic parallel mechanism. Actuation scheme is discussed by covering all the topologies of the metamorphic parallel mechanism. Kong, Gosselin and Richard(15) state a validity condition for actuated joints in parallel mechanisms. Ebrahimi, Carretero and Boudreau(21) present two methods for determining the actuation schemes of a $3-PRRR$ planar manipulator. Matone and Roth(22) relate the location of actuators to the singularity of parallel mechanisms. Literature on actuation scheme is deeper reviewed in Section 4.2.

The importance of correctly evaluating the mobility of mechanisms also arises from the interest in self-aligning mechanism. Self-aligning mechanisms are mechanisms which present no redundant constraints, where redundant constraints can be defined as those constraints whose elimination do not change the mobility of the mechanism.

The kinematic design, *i.e.* the use of exact constraints, is historically attributed to James Clerk Maxwell (23). Later Pollard(24) and Hale and Slocum(25) applied the kinematic design principles to scientific instruments.

Reshetov(26) focuses on the concept of self-alignment for general mechanisms, appointing for the disadvantages of redundant constraints, such as call for higher accuracy in manufacture, increase in weight and size and general reduction of efficiency. Reshetov introduced a method for overconstraint evaluation, based on topology analysis of mechanisms. This method is based on visual inspection on the structure of a mechanism, and does not require any modelling of constraints in terms of screw theory. A new counterexample for this method is presented in Section 5.3.2 as contribution of this Thesis.

A qualitative approach to self-aligning analysis is proposed by Kamm(27) and Blanding(28). They claim the use of kinematic design principles as an essential requirement for the design of instruments and accurate mechanisms. An alternative approach for handling with redundant constraints is proposed by French and Council(29), based on kinematic and elastic design.

Whitney(30) and Shukla and Whitney(31) apply screw theory for determining overconstraint in assembly. Their method permits to evaluate the degree of overconstraint for a class of parallel mechanism. In (32), an extension of this method has been proposed by the author of this Thesis for mobility calculation of parallel mechanism.

The correct evaluation of redundant constraints is of great importance in engineering simulation of multibody systems. Usually redundant constraints are detected and eliminated in order to perform kinematics and dynamics simulations. Wojtyra, Frä et al.(33) propose a method for redundant constraint detection based on the Jacobian formulation. An extension of this method for mechanisms with flexible bodies is presented in (34) and for mechanisms with Coulomb friction in (35). In (36) an algorithm is proposed for the elimination of redundant loop constraints as pre-processing for multibody simulation.

1.2 OBJECTIVES AND METHOD

The focus of this Thesis is on the analysis of overconstrained mechanisms. When the complexity of a mechanism increases in terms of the number of links, joints and loops, qualitative method for overconstraint analysis, as the method proposed by Reshetov(26), are difficult to apply because of the combinatorial explosion of the number of components of the mechanism to be analysed.

Thus the main objective of this Thesis is a new approach for analysis of overconstrained mechanisms in terms of freedoms and constraints, based on screw theory representation and Davies(17) adaptation of Kirchhoff laws to multibody systems. This approach permits modelling freedoms and constraints of a mechanism in terms of vector spaces, thus enabling the use of linear algebra formulation for investigating the main local characteristics of mechanisms.

Based on this formulation, a secondary objective is the analysis of the actuation scheme for a mechanism. Once the degree of freedom and constraint are correctly evaluated, investigation of actuation schemes can be performed with the aim of enumerating all valid schemes. When the complexity of the mechanism increases, the space of solutions grows exponentially, thus methods for enumerating all actuation schemes and selecting an optimal solution is a further objective of this Thesis.

A related problem is the elimination of redundant constraints for a given overconstrained mechanism. Methods for enumeration of all possible self-aligning mechanisms kinematically equivalent to an overconstrained one and selecting an optimal solution are thus a further objective.

Finally, a last objective is a deeper analysis into Reshetov method, based on the formulation of freedoms and constraints introduced in this Thesis, with the aim of formally demonstrating some propositions stated in (26) and verifying the overall validity of the method.

1.3 CONTRIBUTIONS

This Thesis contributes to the analysis of overconstraint mechanism. The adaptation of Kirchhoff's laws to multibody systems proposed by Davies(4) is extended with focus on the state of overconstraint of a mechanism. This formulation of freedoms and constraints permits the introduction of matroid theory for mechanism analysis. A set of new algorithms are thus proposed for investigating actuation schemes of mechanism and self-aligning derived mechanisms. Specific contributions are as follows.

1.3.1 Davies' method

The Davies' method has been extensively used in this work. The formulation originally proposed in (17) has been focused on the state of overconstraint of a mechanism. This approach constitutes the basis for the new results obtained in this Thesis.

- Based on the Kirchhoff laws adaptation to mechanical network, matrix analysis is introduced for overconstraint analysis. This formulation permits loop identification of circuit actions and elimination of redundant constraints. Thus a new method based on the extension of Davies' method is proposed for automatic elimination of redundant constraints in multibody simulations.

1.3.2 Actuation scheme investigation of overconstrained mechanisms

Based on the freedoms and constraints formulations employed for mechanisms, matroid theory is applied for investigating the actuation schemes of mechanisms. More specifically, the following contributions are proposed:

- A new algorithm, based on matroid formulation, is introduced for enumerating all valid actuation schemes for any mechanism. The algorithm has been applied to the study of the *3-PPRR* mechanism, investigated by Zhao et al.(18), and the results compared;

- A new algorithm, based on matroid formulation, is introduced for selecting an optimal actuation scheme with respect to a set of criteria based on mechanism specifications.

1.3.3 Self-aligning mechanisms derived from an overconstrained mechanism

Based on the freedoms and constraints formulations employed for mechanisms, matroid theory is applied for investigating the self-aligning mechanisms kinematically equivalent to an overconstrained mechanism. More specifically, the following contributions are proposed:

- A new algorithm, based on matroid formulation, is introduced for enumerating all possible self-aligning mechanisms kinematically equivalent, *i.e.* with the same topology, mobility, number and dimension of links and number of circuits, to an overconstrained mechanism;

- A new algorithm, based on matroid formulation, is introduced for selecting an optimal self-aligning kinematically equivalent mechanism with respect to a set of criteria based on mechanism specifications.

1.3.4 Reshetov method

The Reshetov method, based on qualitative analysis of the state of overconstraint in mechanisms, is investigated. The following contributions are proposed:

- A formal proof, based on screw theory formulation, is proposed for the conjectures stated in (26).;

- A counterexample for the method has been found and it is introduced;

- The counterexample introduced is analysed, and the flaw in the Reshetov method is investigated by means of screw theory and Davies' formulation.

1.3.5 Mechanism theory contribution

In terms of mechanism theory, the following contribution is proposed:

- A new kinematic chain invariant is introduced, which states the relation between mobilities and redundant constraints in a given overconstrained mechanism.

1.4 THESIS OUTLINE

A review of screw theory is briefly presented in Chapter 2. In the same chapter a literature survey is also presented, with focus on the recent work on mobility calculation.

In Chapter 3 the analysis of mechanisms in terms of overconstraint is presented. The concept of circuit action, introduced by Davies(37), is presented. A closer look insight Davies' equations is then presented, and a new linear algebra approach is introduced to the study of coupling in a mechanism. Based on this approach, linear algebra tools are introduced, resulting in a new method for analysis of overconstrained mechanisms.

In Chapter 4 a brief review of matroids is first presented. Then the linear dependence and independence of freedoms and constraints in a given mechanism are investigated by the application of matroid theory. A new and original approach, based on matroid theory, is then applied for solving two different problems of mechanism: enumeration and selection of valid actuation schemes, enumeration and selection of self-aligning mechanism kinematically equivalent to a given mechanism. A set of new algorithms, original contributions of this Thesis, are finally presented.

In Appendix A a brief review of the main topics of Linear Algebra applied in this Thesis is presented. In Appendix B a deeper review of Matroid Theory is presented, with particular emphasis on the tools employed in this Thesis. Finally, in Appendix C matrices and complementary results are presented for the examples analysed.

2 LITERATURE REVIEW AND NOTATION

In this chapter a brief review of screw theory is presented. The notation for screws is introduced, with focus on overconstraint analysis. Finally, the literature review is presented, with focus on previous works on overconstrained mechanisms.

.

2.1 SCREW THEORY REVIEW

This section provides a concise review of general screw theory, which is extensively used along this Thesis. Screw theory started at the second half of the 19^{th} century from the study on line geometry by Plücker (2). In 1900, the pubblication of the classic work, *A treatise on the Theory of Screws* (1), marked that screw theory is relatively mature. Later many researchers, such as Waldron(38), Hunt(2), Phillips(3) among others, have made important contributions to screw theory.

Geometrically, a screw $\mathbf{\$}$ is a line l together with a scalar pitch h, *i.e.* $\mathbf{\$} = \{l, h\}$. Since the dimension of the space of lines is four, the dimension of the space of screws is five. A screw can represent the instantaneous motion of a rigid body and the instantaneous action that a rigid body is subjected to.

In kinematics, Chasles' theorem or Mozzi-Chasles' theorem states that the most general rigid body displacement can be produced by a translation along a line l (called its screw axis) and by a rotation about the same line.

Thus the instantaneous motion of a rigid body with respects to a fixed frame can be expressed as a rotation around an instantaneous fixed axis plus a translation around the same axis, as shown in Figure 1. Therefore, the instantaneous motion of the rigid body can be represented by a screw $\mathbf{\$}$, called a *motion screw*, and the instantaneous fixed axis is called the screw axis. The ratio between the translation velocity and the angular velocity is called pitch of the screw $h = ||\boldsymbol{\tau}||/||\boldsymbol{\omega}||$.

The instantaneous motion of a rigid body with respect to a fixed frame, when represented by a screw, is expressed by a pair of vectors in the form: $\$ = (\boldsymbol{\omega}, \boldsymbol{V_p})^T$ or in screw coordinates $\mathbf{\$^m} = \begin{bmatrix} r & s & t & u & v & w \end{bmatrix}$. Vector $\boldsymbol{\omega} = \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}^T = \begin{bmatrix} r & s & t \end{bmatrix}^T$ represents the angular velocity of the body with respects to the fixed frame. Vector $\boldsymbol{V_p} = \begin{bmatrix} V_{px} & V_{py} & V_{pz} \end{bmatrix}^T = \begin{bmatrix} u & v & w \end{bmatrix}^T$ represents

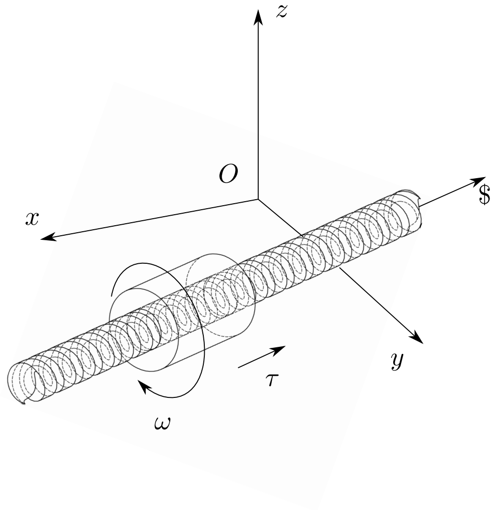


Figure 1 – Instantaneous motion of a rigid body

the linear velocity of a point p moving with the body and instantaneously at the origin O of the fixed frame, as depicted in Figure 2.

Vector $\boldsymbol{V_p}$ has two components: the component parallel to the screw axis $\boldsymbol{\tau} = h\boldsymbol{\omega}$ and the component normal to the screw axis $\boldsymbol{S_O} \times \boldsymbol{\omega}$, where $\boldsymbol{S_O}$ is the position vector of any point on the screw axis with respect to the fixed frame.

Similarly, a force vector $\boldsymbol{Q} = \begin{bmatrix} U & V & W \end{bmatrix}$ and a moment vector $\boldsymbol{P} = \begin{bmatrix} R & S & T \end{bmatrix}$ around the origin represent an action screw in the form $\boldsymbol{\$^a} = \begin{bmatrix} R & S & T & U & V & W \end{bmatrix}$. With these notations the motion screw is said to be written in ray-coordinates and the action screw in axis-coordinates (2).

In general, a combination of linear and angular velocity is called a *twist*, and a combination of torque and force is called a *wrench*. The relation between the first and second part of a twist or a wrench is the pitch h.

A pure force and a pure angular velocity have zero pitch, $h = 0$. A pure torque and a pure linear velocity have infinite pitch, $h \to \infty$. Figure 3 shows the representation of screws with, respectively, zero and infinity pitch.

Sometimes it is convenient to express a screw as a magnitude multiplied by a normalised screw, *i.e.* $\boldsymbol{\$} = \psi\hat{\boldsymbol{\$}}$. The magnitude is $\psi = ||\boldsymbol{\omega}||$, if the motion is pure rotation, $\psi = ||\boldsymbol{V_p}||$ if the motion is pure

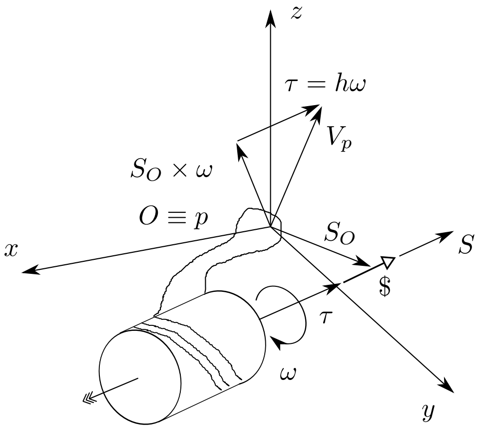


Figure 2 – Screw coordinates

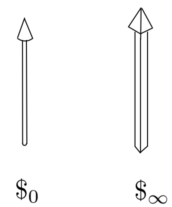


Figure 3 – Representation of screws with pitch $h = 0$ e $h = \infty$

translation. If the motion is a combination of rotation a translation, *i.e.* a twist, $\psi = ||\boldsymbol{\omega}||$. The unit screw can be expressed as a pair of vectors in the form:

$$\hat{\$} = \left[\begin{array}{c} \boldsymbol{S} \\ \boldsymbol{S_O} \times \boldsymbol{S} + h\boldsymbol{S} \end{array} \right] \tag{2.1}$$

where $\boldsymbol{S}$ is the unit vector parallel to the screw axis, with $|\boldsymbol{S}| = 1$. Generally, the magnitude of a particular screw aligned with one of the canonical axes is denoted by the screw component. For example a pure rotation along the z axis can be written as a magnitude multiplying a unit screw in the form:

$$t\hat{\$}^m = t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.2}$$

where t is the component of a unit screw aligned along the axis z. In the same way a pure force along x axis can be written as:

$$U\hat{\$}^a = U \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{2.3}$$

Given two screws, $\boldsymbol{\$^1}$ and $\boldsymbol{\$^2}$, their scalar product is denoted as $\boldsymbol{\$^1} \circ \boldsymbol{\$^2}$. When the scalar product of two screw is zero, $\boldsymbol{\$^1} \circ \boldsymbol{\$^2} = 0$, the two screws are said reciprocal. The scalar product of the instantaneous action $\boldsymbol{\$^a}$ acting on a rigid body and of the motion $\boldsymbol{\$^m}$ to which the rigid body is subjected can be regarded as the power expended by the action screw $\boldsymbol{\$^a}$ on the motion screw $\boldsymbol{\$^m}$. Using the notation introduced above, the power expended by wrench $\boldsymbol{\$^a}$ on a twist $\boldsymbol{\$^m}$ is:

$$\mathcal{P} = \boldsymbol{\$^a} \circ \boldsymbol{\$^m} = rR + sS + tT + uU + vV + wW \tag{2.4}$$

which can easily be remembered as the scalar product.

The concept of a *screw system* is deduced from kinematics. For an open chain or a serial robot, the motion of the end-effector can be expressed as the summation of the motions of all links. When the motions of all the links of the serial chain are expressed as screws, the motion of the end-effector is the linear combination of all screws. All the screws which determine the motions of all the links of a serial chain form a screw system. More generally, when all the kinematic pairs of a mechanism are expressed as screws, all the screws in the mechanism generate a screw system (13).

According to the number of independent screws, screw systems can be divided into n-order screw systems with $n = 1, \ldots, 6$. According to the properties of reciprocal screws, 4-order and 5-order screw systems can be transformed into 2-order and 1-order corresponding reciprocal screw systems to investigate their principal screws.

Traditionally, the principal screws of a screw system are defined as the screws whose pitches are the extremes and equal the order of the system in number (12). Any screw in a screw system can be expressed as a linear combination of a set of maximal linearly independent screws of the system. Therefore, the maximal linearly independent screws in a screw system can be utilized to completely represent the system. Theoretically, any set of such independent screws is enough to determine the screw system. For a specified screw system, the maximum and minimum pitches of the screws describe the characteristics of the system.

A complete classification of the main screw system can be found in (2), (3) and (39). For example, the three-system is composed by three linearly independent screws $\boldsymbol{\$_\alpha}$, $\boldsymbol{\$_\beta}$ and $\boldsymbol{\$_\gamma}$, that is, not all belonging to the same two system. According to the principal screws pitches h_α, h_β and h_γ, the three-order screw system is divided into ten special three-systems (2). In the fourth special three-system the pitches of the principal screws are respectively $h_\alpha = \infty$, $h_\beta = h_\gamma$ (both finite). For example, the fourth special three-system describes the planar actions to which a planar mechanism is subjected, *i.e.* a moment $(h \to \infty)$ and two forces $(h = 0)$. In the fifth special three-system the pitches of the principal screws are respectively $h_\alpha = h_\beta = \infty$, h_γ finite. For example, the fifth special three-system describes the planar motions of a planar mechanism, *i.e.* two linear velocities $(h \to \infty)$ and an angular velocity $(h = 0)$.

In terms of linear algebra, a screw system $\mathbf{\Delta}$ is a linear vector subspace with dimension $\lambda = dim(\mathbf{\Delta})$ (14). The screw system $\bar{\Delta}$ that is reciprocal to $\mathbf{\Delta}$ is defined by:

$$\bar{\mathbf{\Delta}} \equiv \{\boldsymbol{\$_1} | \boldsymbol{\$_1} \circ \boldsymbol{\$_2} = 0, \forall \boldsymbol{\$_2} \in \mathbf{\Delta}\} \tag{2.5}$$

The dimension of a screw system and its reciprocal are related by:

$$dim\left(\mathbf{\Delta}\right) + dim\left(\bar{\mathbf{\Delta}}\right) = 6 \tag{2.6}$$

The minimum order of the screw system to which all motion and action screws under consideration belong is the dimension λ with $1 \leq \lambda \leq 6$.

2.1.1 Free motions and constraints of a kinematic pair

Generally, a kinematic pair reduces the freedoms between two links of a mechanism. This is not always true for parallel mechanisms, as it will be analysed in details in Section 2.2.2. A motion screw $\$^m$ and an action screw $\$^a$ can be associated with each joint, describing respectively the motions allowed and the actions transmitted by the kinematic pair. The helical pair in Figure 4 is considered as example. It owns one degree of freedom, which can be described by the motion screw associated with the joint, with respect to the Cartesian coordinate system:

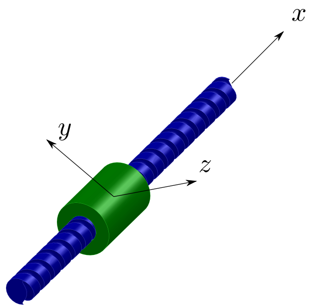


Figure 4 – Helical Pair

$$\hat{\$}_h^{\boldsymbol{m}} = \begin{bmatrix} 1 & 0 & 0 & h & 0 & 0 \end{bmatrix}^T \tag{2.7}$$

The constraints transmitted by the helical pair can be expressed in terms of the action screws associated:

$$\begin{array}{ll} \hat{\$}_{h1}^{\boldsymbol{a}} = \begin{bmatrix} -h & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T, & \text{a wrench constraint along the } x\text{-axis with a negative pitch } -h\text{: a moment constraint about the } x\text{-axis plus a force along the } x\text{-axis;} \\ \hat{\$}_{h2}^{\boldsymbol{a}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T, & \text{a force constraint along the } y\text{-axis;} \\ \hat{\$}_{h3}^{\boldsymbol{a}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T, & \text{a force constraint along the } z\text{-axis;} \\ \hat{\$}_{h4}^{\boldsymbol{a}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T, & \text{a moment constraint about the } y\text{-axis;} \\ \hat{\$}_{h5}^{\boldsymbol{a}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T, & \text{a moment constraint about the } z\text{-axis;} \end{array} \tag{2.8}$$

The main kinematic pairs, with the motion and action screws associated, are presented in Tables (1) and (2).

Table 1 – Kinematic pairs I

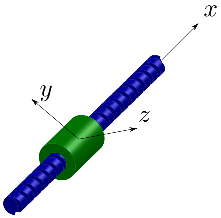
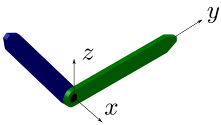
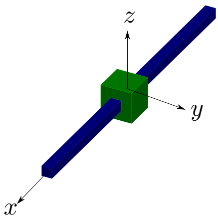
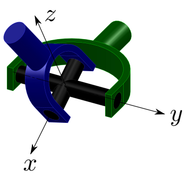
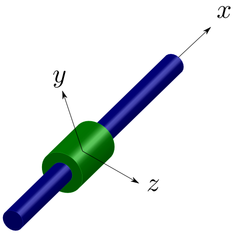
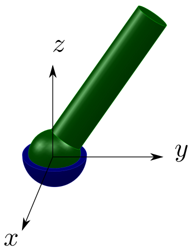
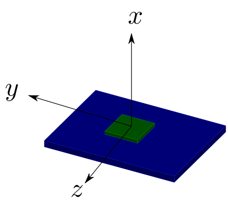
Kinematic Pair	**Motion and Action Screws**
 **Helical Pair**	$\hat{\$}_h^{\boldsymbol{m}} = \begin{bmatrix} 1 & 0 & 0 & h & 0 & 0 \end{bmatrix}^T$ $\hat{\$}_{h1}^{\boldsymbol{a}} = \begin{bmatrix} -h & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T$ $\hat{\$}_{h2}^{\boldsymbol{a}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T$ $\hat{\$}_{h3}^{\boldsymbol{a}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$ $\hat{\$}_{h4}^{\boldsymbol{a}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$ $\hat{\$}_{h5}^{\boldsymbol{a}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T$
 **Revolute Pair**	$\hat{\$}_r^{\boldsymbol{m}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$ $\hat{\$}_{r1}^{\boldsymbol{a}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T$ $\hat{\$}_{r2}^{\boldsymbol{a}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T$ $\hat{\$}_{r3}^{\boldsymbol{a}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$ $\hat{\$}_{r4}^{\boldsymbol{a}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$ $\hat{\$}_{r5}^{\boldsymbol{a}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T$
 **Prismatic Pair**	$\hat{\$}_p^{\boldsymbol{m}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T$ $\hat{\$}_{p1}^{\boldsymbol{a}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T$ $\hat{\$}_{p2}^{\boldsymbol{a}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$ $\hat{\$}_{p3}^{\boldsymbol{a}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$ $\hat{\$}_{p4}^{\boldsymbol{a}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$ $\hat{\$}_{p5}^{\boldsymbol{a}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T$
 **Universal Pair**	$\hat{\$}_{u1}^{\boldsymbol{m}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$ $\hat{\$}_{u2}^{\boldsymbol{m}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$ $\hat{\$}_{u1}^{\boldsymbol{a}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T$ $\hat{\$}_{u2}^{\boldsymbol{a}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T$ $\hat{\$}_{u3}^{\boldsymbol{a}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$ $\hat{\$}_{u4}^{\boldsymbol{a}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T$

Table 2 – Kinematic pairs II

Kinematic Pair	**Motion and Action Screws**
 **Cylindrical Pair**	$\hat{\$}^{\boldsymbol{m}}_{c1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$ $\hat{\$}^{\boldsymbol{m}}_{c2} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T$ $\hat{\$}^{\boldsymbol{a}}_{c1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T$ $\hat{\$}^{\boldsymbol{a}}_{c2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$ $\hat{\$}^{\boldsymbol{a}}_{c3} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$ $\hat{\$}^{\boldsymbol{a}}_{c4} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T$
 **Spherical Pair**	$\hat{\$}^{\boldsymbol{m}}_{s1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$ $\hat{\$}^{\boldsymbol{m}}_{s2} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$ $\hat{\$}^{\boldsymbol{m}}_{s3} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T$ $\hat{\$}^{\boldsymbol{a}}_{s1} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T$ $\hat{\$}^{\boldsymbol{a}}_{s2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T$ $\hat{\$}^{\boldsymbol{a}}_{s3} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$
 **Planar Pair**	$\hat{\$}^{\boldsymbol{m}}_{pl1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$ $\hat{\$}^{\boldsymbol{m}}_{pl2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T$ $\hat{\$}^{\boldsymbol{m}}_{pl3} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$ $\hat{\$}^{\boldsymbol{a}}_{pl1} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T$ $\hat{\$}^{\boldsymbol{a}}_{pl2} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$ $\hat{\$}^{\boldsymbol{a}}_{pl3} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T$

2.2 LITERATURE REVIEW

In this section, a brief review of the previous works about over-constrained mechanisms is presented.

2.2.1 Davies' method

In this section an overview of Davies' method is briefly presented. A deeper insight into Davies' formulation is introduced in Chapter 3.

The main contributions introduced by Davies and its applications can be found in (17) (40) (16) (41) (42) (4) (43)(44) (45).

Davies adapted Kirchhoff's circulation law and cutset law to multibody systems. The adaptation of Kirchhoff's laws is based on the representation of a coupling network(41) with n links and g joints by a graph, called a coupling graph G_C, in which every link (body) is represented by a node and every direct coupling between links by an edge. In Figure 5a a four-bar mechanism is presented. Its structural representation is shown in Figure 5b and the coupling graph G_C in Figure 5c.

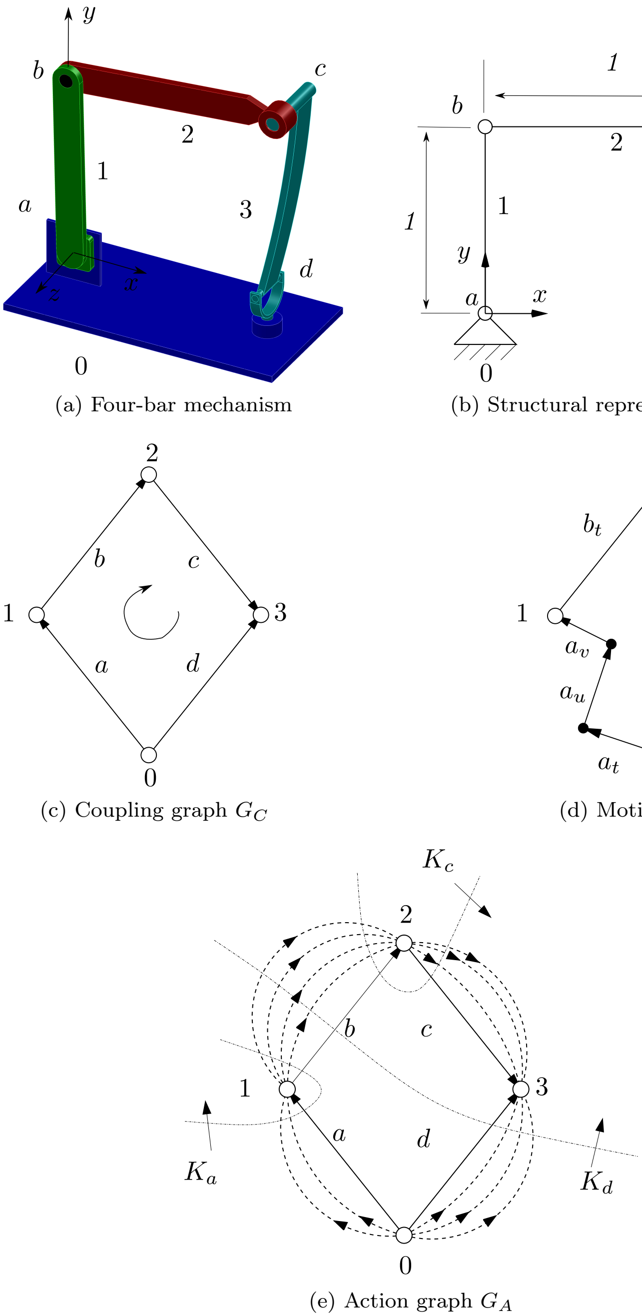


Figure 5 – Four-bar mechanism (a), structural representation (b), coupling graph G_C (c), motion graph G_M (d) and action graph G_A (e)

From coupling graph G_C the motion graph G_M is obtained: an edge of G_C that represents a direct coupling i of freedom f_i is replaced in G_M by f_i edges in series, representing a set of independent motions that together span the f_i-system motion screws of the coupling. Thus the graph G_M contains F edges, where $F = \sum_1^g f_i$ is the gross dof of the coupling network, *i.e.* the sum of the dof f_i of the couplings of the kinematic chain. In the same way a new graph, the action graph G_A, can be defined from G_C, where each edge i of G_C is replaced by c_i edges in parallel, with c_i being the degree of constraint *doc* of edge i.

The mechanism of Figure 5a is considered. As it is a spatial mechanism, *i.e.* $\lambda = 6$, for each joint i $f_i + c_i = 6$. Thus, the planar coupling a allows $f_a = 3$ freedoms and imposes $c_a = 3$. Coupling b is a revolute one, and it allows $f_b = 1$ degree of freedom and imposes $c_a = 5$ constraints. Couplings c and d are respectively a cylindrical pair and a universal pair, each of one allowing two degrees of freedom, *i.e.* $f_c = 2$ and $f_d = 2$, and imposing each one $c_c = 4$ and $c_d = 4$ constraints, respectively. Thus joint a can be replaced by $f_a = 3$ couplings in series, joint c and d by two couplings in series. In Figure 5d the graph G_M is presented, where each edge i of graph G_C has been replaced by f_i coupling in series. In the same way graph G_A is presented in Figure 5e.

Two sets of variables are associated with each joint: the variables which represent the motions allowed by the coupling and the variables which represent the actions transmitted by the coupling. These variables differ from their electrical counterparts in two aspects. Firstly, every motion and every action is modeled as a geometric screw, requiring λ coordinates with $1 \leq \lambda \leq 6$, where λ is the minimum order of the screw system to which all motion and action screws belong. Secondly, a coupling can transmit/allow up to λ independent actions/motions.

Assuming the motions allowed by the couplings, a motion screw $\$^m$ can be associated to each of the f_i freedom of joint i, *i.e.* to each of the edge of graph G_M. Thus the adaptation of Kirchhoff's circulation law permits finding a set of independent instantaneous screws associated with the given kinematic chain. It requires that for any closed sequence of bodies in relative motion, each of the λ motion coordinates for the bodies belonging to the loop sums to zero. In general, for a single circuit of couplings and dimension λ of six, it can be written as:

$$\sum r = \sum s = \sum t = \sum u = \sum v = \sum w = 0 \tag{2.9}$$

where r, s, t are the angular velocity components and u, v, z are the linear velocity components of motion screws.

Given a graph with g edges and n links, every spanning tree T defines $g - n + 1$ fundamental circuits in the form $T + c_i$, where c_i is a chord of the graph (46). Thus, recalling the graph representation of a mechanism previously introduced, the concept of fundamental circuits can be extended to mechanisms.

There are λ equations for each circuit, so for a coupling network having ν fundamental circuits the circuit law can be arranged as:

$$\left[\boldsymbol{\hat{M}_N}\right]_{\lambda\nu,F}[\boldsymbol{\psi}]_F = [\mathbf{0}]_{\lambda\nu} \tag{2.10}$$

where $[\boldsymbol{\psi}]_F$ is the vector of F generalised motion magnitudes in the form:

$$[\boldsymbol{\psi}]_{F,1} = \begin{bmatrix} \psi_{a_1} & \vdots & \psi_{a_i} & \cdots & \psi_{b_1} & \vdots & \cdots & \psi_F \end{bmatrix}^T_{F,1} \tag{2.11}$$

with ψ_i being an angular velocity component r, s, t or a linear velocity component u, v, w, and

$$\left[\boldsymbol{\hat{M}_N}\right]_{\lambda\nu,F} = \begin{bmatrix} \left[\boldsymbol{\hat{M}_D}\right]_{\lambda,F}[\boldsymbol{B_1}]_{F,F} \\ \left[\boldsymbol{\hat{M}_D}\right]_{\lambda,F}[\boldsymbol{B_2}]_{F,F} \\ \cdots \\ \left[\boldsymbol{\hat{M}_D}\right]_{\lambda,F}[\boldsymbol{B_\nu}]_{F,F} \end{bmatrix}_{\lambda\nu,F} \tag{2.12}$$

is the network unit motion matrix of the coupling network. $\boldsymbol{\hat{M}_D}$ is the unit motion matrix which contains one unit motion screw for column:

$$\left[\boldsymbol{\hat{M}_D}\right]_{\lambda,F} = \begin{bmatrix} \hat{\$}^m_{a_1} & \hat{\$}^m_{a_i} & \vdots & \hat{\$}^m_{b_1} & \vdots & \ldots & \vdots & \hat{\$}^m_F \end{bmatrix} \tag{2.13}$$

with $\hat{\$}^m_i$ is the unit motion screw representing a single allowed motion of coupling i, *i.e.* an edge of motion graph G_M. $[\boldsymbol{B_i}]_{F,F}$ with $i = 1, 2, \cdots, \nu$ are diagonal matrices with diagonal elements corresponding to row i of the circuit matrix $[\boldsymbol{B_M}]_{\nu,F}$ (17).

The mechanism presented in Figure 5a has on single loop and it can be verified that for single-loop mechanism $\hat{\boldsymbol{M}}_{\boldsymbol{N}} = \hat{\boldsymbol{M}}_{\boldsymbol{D}}$. Thus for this mechanism Equation (2.10) can be written as:

$$\begin{array}{c} \begin{array}{cccccccc} a_t & a_u & a_v & b_t & c_t & c_w & d_s & d_t \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}_{6,8} \end{array} \begin{bmatrix} t_a \\ u_a \\ v_a \\ t_b \\ t_c \\ w_c \\ s_d \\ t_d \end{bmatrix}_{8,1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{6,1} \tag{2.14}$$

In Equation (2.14) coupling a is a planar pair, thus it allows $f_a = 3$ motions. Each one of the columns a_r, a_u, a_v of matrix $\hat{\boldsymbol{M}}_{\boldsymbol{D}}$ describes an unit motion screw corresponding to one motion allowed by coupling a. It is important to assume that all motion screws components of couplings a, b, c and d are expressed in the coordinate system represented in Figures 5a and 5b. Thus, the revolute coupling b motion screws can be written in the coordinate system as:

$$\$_b^m = \begin{bmatrix} 0 \\ 0 \\ \omega_b \\ b_y\omega_b \\ -b_x\omega_b \\ 0 \end{bmatrix} \tag{2.15}$$

where b_x and b_y are the coordinates of joint b. The remaining columns of matrix $\hat{\boldsymbol{M}}_{\boldsymbol{D}}$ describes the unit motion screws corresponding to the motions allowed respectively by coupling c and d in the coordinate system. The vector $\boldsymbol{\psi}$ represents the motions screws unknown magnitudes of $\hat{\boldsymbol{M}}_{\boldsymbol{D}}$.

Equation (2.14) can be thus regarded as an linear homogeneous system with λ equations in $F = 8$ unknowns. Therefore, the solution of system (2.14) identifies which, if any, of the F unknowns are zero and provides expressions for the remainder in terms of F_N primary variables. The set of primary variables describes the state of freedom of the mechanism, and the number of primary variables F_N is the *degree of freedom*, or *mobility*, of the mechanism (17). In Appendix C.1 Davies' method is applied to a four-bar mechanism in order to calculate the

instantaneous kinematics. An example of Davies' method for a multi-loop mechanism in Appendix C.5.
On the other hand, the adaptation of Kirchhoff's cutset law requires that for any network of coupled bodies in equilibrium, wherever there exists a cutset (46) of couplings, each of the couplings λ action coordinates sums to zero. In general, given a spatial mechanism with $\lambda = 6$, for each cutset , it can be written as:

$$\sum R = \sum S = \sum T = \sum U = \sum V = \sum W = 0 \tag{2.16}$$

where R, S, T are the moment components and U, V, W are the force components imposed by the couplings.

For a coupling network the cutset law can be written as:

$$\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]_{\lambda k, C} [\boldsymbol{\Psi}]_C = [\mathbf{0}]_{\lambda k} \tag{2.17}$$

where k is the number of the cutsets in the coupling network, $C = \sum_{1}^{g} c_i$ is the gross degree of constraint, *i.e.* the sum of the degree of constraint c_i imposed by the couplings and $\boldsymbol{\Psi}$ is the vector of the action screws unknown magnitudes imposed by the couplings. Matrix $\hat{\boldsymbol{A}}_{\boldsymbol{N}}$ can be written as:

$$\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]_{\lambda k, C} = \begin{bmatrix} \left[\hat{\boldsymbol{A}}_{\boldsymbol{D}}\right]_{\lambda, C} & [\boldsymbol{Q}_{\boldsymbol{1}}]_{C,C} \\ \left[\hat{\boldsymbol{A}}_{\boldsymbol{D}}\right]_{\lambda, C} & [\boldsymbol{Q}_{\boldsymbol{2}}]_{C,C} \\ \vdots & \vdots \\ \left[\hat{\boldsymbol{A}}_{\boldsymbol{D}}\right]_{\lambda, C} & [\boldsymbol{Q}_{\boldsymbol{K}}]_{C,C} \end{bmatrix} \tag{2.18}$$

where $[\boldsymbol{Q}_{\boldsymbol{i}}]_{C,C}$, $i = 1, 2, \ldots, k$ are diagonal matrices whose diagonal elements correspond to row i of cutset matrix $[\boldsymbol{Q}_{\boldsymbol{A}}]_{k,C}$, derived from action graph G_A.
$\left[\hat{\boldsymbol{A}}_{\boldsymbol{D}}\right]$ is the unit action matrix containing one action screw for column:

$$\left[\hat{\boldsymbol{A}_{\boldsymbol{D}}}\right] = \left[\ \overbrace{\$^a_{a1} \quad \$^a_{a2} \quad \ldots}^{\text{Coupling a}} \ \vdots\ \overbrace{\$^a_{b1} \quad \ldots}^{\text{Coupling b}} \ \vdots\ \overbrace{\ldots}^{\ldots} \ \vdots\ \overbrace{\ldots \quad \$^a_C}^{\ldots}\ \right] \tag{2.19}$$

where k is the number of the cutsets in the coupling network, λ is the order of the screw system to which all action screws belong (in the most general case $\lambda = 6$) and $C = \sum c_i$ is the gross degree of constraint, *i.e.* the sum of the degree of constraint c_i of the couplings of the kinematic chain. Equation (2.19) is the dual to Equation (2.10) (17).

Considering mechanism presented in Figure 5a and its action graph G_A, presented in Figure (e), the unit action matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{D}}\right]$ can be written as:

$$\left[\hat{\boldsymbol{A}}_{\boldsymbol{D}}\right] = \begin{array}{c} \begin{array}{ccc|ccccc|cccc|cccc} a_R & a_S & a_w & b_R & b_S & b_U & b_V & b_W & c_R & c_S & c_U & c_V & d_R & d_U & d_V & d_W \end{array} \\ \left[\begin{array}{ccc|ccccc|cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]_{6,16} \end{array} \tag{2.20}$$

and cutset matrix $\hat{\boldsymbol{Q}}_{\boldsymbol{A}}$ can be written as:

$$[\boldsymbol{Q}_{\boldsymbol{A}}]_{3,20} = \begin{array}{c} \begin{array}{ccc|ccccc|cccc|cccc} a_R & a_S & a_W & b_R & b_S & b_U & b_V & b_W & c_R & c_S & c_U & c_V & d_R & d_U & d_V & d_W \end{array} \\ \begin{array}{c} Q_1 \\ Q_2 \\ Q_3 \end{array} \left[\begin{array}{ccc|ccccc|cccc|cccc} 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right] \end{array} \tag{2.21}$$

Thus for this mechanism Equation (2.17) can be written as:

$$
\begin{array}{c}
\begin{array}{ccc|ccccc|cccc|cccc}
a_R & a_S & a_w & b_R & b_S & b_U & b_V & b_W & c_R & c_S & c_U & c_V & d_R & d_U & d_V & d_W
\end{array} \\
\left[\begin{array}{ccc|ccccc|cccc|cccc}
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & -0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]_{18,16}
\end{array}
\begin{bmatrix} R_a \\ S_a \\ W_a \\ R_b \\ S_b \\ U_b \\ V_b \\ W_b \\ R_c \\ S_c \\ U_c \\ V_c \\ R_d \\ U_d \\ V_d \\ W_d \end{bmatrix}_{16,1}
= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T_{18,1}
\tag{2.22}
$$

Thus, Equation (2.22) can be regarded as an linear homogeneous system with λk equations in C unknowns. Therefore, the solution of system (2.22) identifies which, if any, of the C unknowns are zero and provides expressions for the remainder in terms of C_N primary variables, where C_N is defined by Davies as the net degree of constraint of the coupling network. In terms of linear algebra, the C_N primary variables are called *free variables*. Once given arbitrary values to the free variables, the values of the other unknowns of the linear homogeneous system can be deduced by *back-substitution*, as described in Appendix A. In this Thesis, C_N is refereed as the degree of redundant constraint, as it can be regarded as the number of redundant constraint of the coupling network. The cutset law formulated by Davies is analysed in greater details in the next chapters.

Based on this formulation, the following variables in a kinematic chain can be calculated: motions and actions (17), passivity and redundancy of mobile and immobile mechanical networks (47), formulae for the degrees of freedom and degree of redundancy (48); and network actions (37).

In (17), two more methods are described that perform the same task as the adaptation of Kirchhoff's laws, based on the principle of virtual power. In this Thesis the model and analysis of redundant constraints are derived from the circuit actions and virtual motions analysis introduced by Davies in (17, 48).

2.2.1.1 Applications of Davies method

Davies' method has been employed by several authors for different applications. A brief review of the main application is herein presented.

In order to analyse gear trains, Cazangi and Martins(49) employ this method for the analysis of two gear trains; one has two degrees of freedom, two forward ratios and one backward; the second has three degrees of freedom, three forward ratios and one backward. Laus, Simas and Martins(45) employ the adaptation of Kirchhoff's equations for studies of the efficiency of an epicyclic gear train and machines in general, including parallel robots. For both, account is taken of friction, including gear tooth friction. Tischler, Lucas and Samuel(50) employs this method for a study of friction in multi-loop linkages.

Assuming kinematic chains in critical configurations, Tischler(51) applyies in the study of critical configurations of a RCCC kinematic chain. Davies and Laus(44) do likewise for a planar 6-Link Stephenson kinematic chain.

Finally, the Davies' method has been extensively employed in combination with the use of virtual couplings (Assur groups). An Assur group does not introduce additional constraints. For example, for a planar manipulator it can comprise PPR couplings in series; for a spatial manipulator PPPRRR or PPPS couplings in series. This approach revealed very useful as the primary variables can be either those of couplings of the manipulator or, for inverse kinematics, couplings of the Assur group. Erthal, Nicolazzi and Martins(52) use them for a study of vehicle suspension; Campos, Gunther and Martins(43) for the inverse kinematics of serial manipulators and (53) for the inverse kinematics of parallel manipulators. Inverse kinematics also gets attention from Simas et al.(54).

Regarding the study of underwater manipulation, there is the work of Guenther et al.(55). Simas et al.(56) and Rocha et al.(57) applied the method to avoid collisions and for carrying out tasks such as remote repair. Ribeiro and Martins(58) describe the use of virtual chains in studies of cooperating robots. Recently, Saldias et al.(59) extended the application of Davies' method and Assur groups to the modelling of the human knee to aid pre-operative planning.

2.2.2 Review of overconstraint mechanism analysis

In this section a brief review of further contributions found in literature, apart from the Davies' method, about overconstraint mechanism analysis is presented. More detailed literature review is presented along this Thesis.

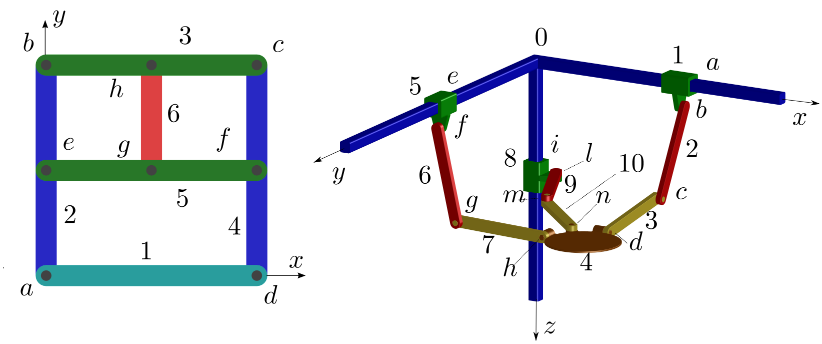
The IFToMM (60) reports the following definition of mobility:

Definition 1. *Number of independent coordinates needed to define the configuration of a kinematic chain or mechanism*

Although the definition of mobility is a concept well established, its calculation has been deeply investigated for more than a century. The most well known mobility criterion, the Grübler-Kutzbach formulation (7), can be written as:

$$F_N = \lambda(n - g - 1) + \sum_{i=1}^{j} f_i \tag{2.23}$$

where λ is the order of the screw system to which all screws belong, n is the number of links, g is the number of joints and f_i is the number of freedom allowed by joint i in the mechanism. Equation (2.23) can be successfully used in analysing almost all planar and some spatial mechanisms. However it fails to analyse more complex overconstrained mechanisms. Two counterexamples for the Grübler-Kutzbach criterion are presented in Figure 6.



(a) (b)

Figure 6 – Counterexamples for Grübler-Kutzbach criterion.

The planar mechanism in Figure 6a has $n = 6$ links, $g = 8$ single-freedom joints and $\lambda = 3$ as it is planar. Thus applying Equation (2.23)

$F_N = 3(6-8-1)+8 = -1$, which is incorrect. In fact the mechanism has $F_N = 1$ degree of freedom. The spatial mechanism in Figure 6b has 11 links, $g = 12$ single-freedom joints and $\lambda = 6$ as it is spatial. Thus applying Equation (2.23) $F_N = 6(11-12-1)+12 = 0$, which is incorrect. In fact the mechanism has $F_N = 3$ degrees of freedom.

A closer insight into the Grübler-Kutzbach formulation can explain the reason of the incorrect results. In Equation (2.23) g is the number of joints of the mechanism, with the assumption that each joint retires some freedom from the mechanism (3). Regrading that a redundant constraint is defined as a constraint whose removal does not alter the mobility of the mechanism (26), it can be observed that the number C_N of redundant constraints present in the mechanism affects the mobility calculation.

Several authors have proposed different forms of mobility calculation. A critical review is presented in (8), where the most important contributions to mobility calculation are analysed and limitations and counterexamples reported.

More recently Huang *et al.* (61, 62, 13) introduced and demonstrated the *Modified Grübler-Kutzbach Criterion* in the form:

$$F_N = \lambda(n-g-1) + \sum_{i=1}^{j} f_i + C_N \tag{2.24}$$

where C_N is the number of redundant constraints. Equation (2.24) has been further theoretically demonstrated by Dai, Huang and Lipkin(14) and applied to several parallel mechanisms, correctly calculating the mobility. In (63) the Modified Grübler-Kutzbach Criterion has been successfully applied to almost all classical mechanism. Finally Zeng, Lu and Huang(64) provides a theoretical basis for the calculation of mobility for the general multi-loop spatial mechanisms, applying Equation (2.24) and equivalent forms.

In (12), (13) and (14) the theory of reciprocal screw is extensively employed in order to determine the number of redundant constraints and correctly evaluate the mobility of parallel mechanism through the Modified Grübler-Kutzbach Criterion.

It should be noted that the different methods for mobility analysis proposed in the cited works are based on the fact that the mobility of a mechanism equals the total number of degrees-of-freedom of links minus the sum of constraints produced by all kinematic pairs, and then plus the number of redundant constraints. Therefore, these formulas have no essential differences although they appear in different forms, and most of them can even be transformed easily from one to another.

However, the big and essential difference among them is the process of identifying the redundant constraints.

The degree of freedom of a mechanism is of primary importance for investigating the motions and actuations of mechanism. Moreover, the number of actuators needed to control the whole mechanism is equal to the mobility of the mechanism (12). An important distinction must be made regarding the mobility of the mechanism. The end effector of a mechanism has been previously defined as the output link of the mechanism (10), *i.e.* the part designed to interact with the environment. The *dof* of the an effector is no larger than 6, *i.e.* the freedoms of a body in space, but the number of independent actuators required to uniquely control the end effector might be any non-negative integer. Two definitions are thus introduced by Zhao et al.(12):

Definition 2. *The degree of freedom of an end effector totally characterizes the motions of the end effector including the number, type and direction of the independent motions and it is referred by Zhao et al.(12) as DOF, with DOF≤ 6 .*

Definition 3. *The degree of freedom of a mechanism with an end effector,* i.e. *the mobility of the whole mechanism, indicates the independent number of actuations required to uniquely control the end effector under any configurations and it is referred by Zhao et al.(12) as CDOF.*

The 9-bar planar mechanism showed in Figure 7, where the end effector is assigned to link $e-f$, is considered as an example. The DOF of the mechanism, *i.e.* the number of the independent motions of end effector $e-f$ is $DOF = 3$, as the mechanism is planar. On the other hand, the $CDOF$ of the mechanism, *i.e.* the number of independent actuators needed to uniquely control the full mechanism is $CDOF = 6$.

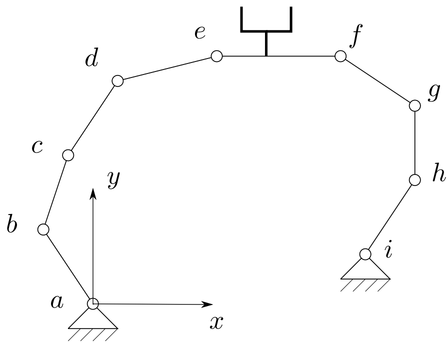


Figure 7 – 9-bar planar mechanism

Along this Thesis, the mobility of the whole mechanism is investigated, referred as the *dof* degree of freedom of the mechanism and indicated as F_N. On the other hand the redundant degree of constraint of the whole mechanism is indicated C_N.

3 REDUNDANT CONSTRAINTS ANALYSIS

In this chapter the analysis of mechanisms in terms of overconstraint is presented. The concept of circuit action is first introduced, based on the work of Davies(17). A closer look insight Davies' equations is then presented, and a new linear algebra approach is introduced to the study of coupling in a mechanism. Based on this approach, linear algebra tools are applied to the network unit motion matrix $\left[\hat{\boldsymbol{M}}_{\boldsymbol{D}}\right]$ and the network unit action matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{D}}\right]$, resulting in a new method for analysis of overconstrained mechanisms.

Based on this modelling, a further analysis of a mechanism is performed in Chapter 4, by means of matroid theory.

3.1 CIRCUIT ACTIONS AND REDUNDANT CONSTRAINTS

Redundant constraints are generally defined as constraints that can be removed without changing the mobility of the system. A redundant constraint in a kinematic chain is always associated with a loop, or a set of loops. On the other hand a serial kinematic chain, as the one presented in Figure 8, does not have any redundant constraints. For example, link 1 is constrained by base 0 through only coupling a and applies a single constraint to link 2 through coupling b. The same holds for the all links of a serial chain.

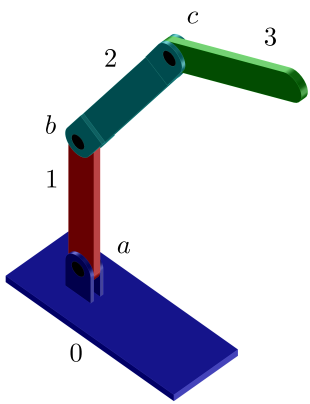


Figure 8 – Serial mechanism with no redundant constraints

When one or more redundant constraints are present in a closed kinematic chain, an action can be locked inside the respective loop of the kinematic chain. This action has been called circuit action by Davies(37) and can be modelled as a wrench on a screw as, in the most general case, the pitch of the screw will not be either exactly zero or tend to infinite.

An example, shown in Figure 9, can provide a better understanding of the concept of circuit action.

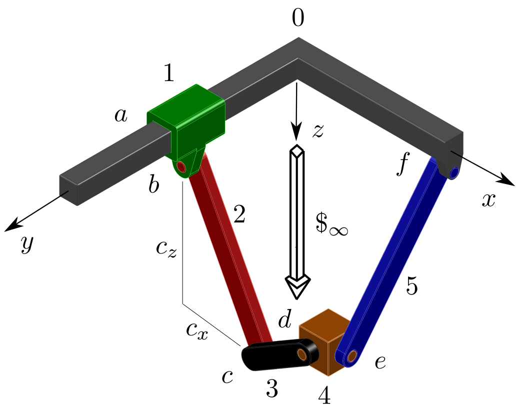


Figure 9 – Single-loop spatial mechanism with redundant constraint, $F_N = 1$ and $C_N = 1$.

The mechanism of Figure 9 is a single-loop spatial mechanism, with mobility $F_N = 1$ and one redundant constraint $C_N = 1$. It has $n = 6$ links and $g = 6$ joints. Joint a is a prismatic coupling pair along y-axis, joints b, c and d are revolute couplings along the same axis, and joints e and f are revolute couplings along x-axis.

The single redundant constraint can be detected by visual inspection, regarding the constraints applied by limbs $a{-}1{-}b{-}2{-}c{-}3{-}d$ and $f - 5 - e$ between links 0 and 4. Both legs constraint the same degree of freedom, *i.e.* the rotation of link 4 around axis-z, thus introducing a redundant constraint in the mechanism.

The same result can be obtained applying screw theory, using the method proposed in (14), (12) and (13). The motions screws associated with the couplings are respectively:

$$\begin{aligned}
\hat{\$}_a^m &= \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]^T \\
\hat{\$}_b^m &= \left[\begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right]^T \\
\hat{\$}_c^m &= \left[\begin{array}{cccccc} 0 & 1 & 0 & -c_z & 0 & c_x \end{array}\right]^T \\
\hat{\$}_d^m &= \left[\begin{array}{cccccc} 0 & 1 & 0 & -d_z & 0 & d_x \end{array}\right]^T \\
\hat{\$}_e^m &= \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & -e_z & -e_y \end{array}\right]^T \\
\hat{\$}_f^m &= \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right]^T
\end{aligned} \tag{3.1}$$

where c_x, c_z, d_x, d_z, e_y and e_z are the coordinates of the joints in the coordinate system. In Figure 9 only coordinates c_x and c_z are showed as example. It is important to regard that all motions screws associated with the couplings are represented in the same coordinate system indicated in Figure 9. In this coordinate system the motion screw associated with joint c can be written as:

$$\hat{\$}_{\boldsymbol{b}}^{\boldsymbol{m}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -c_z \\ 0 \\ c_x \end{bmatrix} \tag{3.2}$$

where the terms $-c_z$ and c_x represent the linear velocity components of the motion screw, as defined in Equation (2.1).

Thus the motion screws associated with limbs a–1–b–2–c–3–d and $f-5-e$ are respectively:

$$\begin{aligned}
\left\{\hat{\$}_{abcd}^m\right\} &= \left[\begin{array}{cccc} \hat{\$}_a^m & \hat{\$}_b^m & \hat{\$}_c^m & \hat{\$}_d^m \end{array}\right]^T \\
\left\{\hat{\$}_{ef}^m\right\} &= \left[\begin{array}{cc} \hat{\$}_e^m & \hat{\$}_f^m \end{array}\right]^T
\end{aligned} \tag{3.3}$$

and the reciprocal action screws are:

$$\left\{\hat{\$}_{abcd}^a\right\} = \left\{ \begin{array}{c} \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right]^T \\ \left[\begin{array}{cccccc} 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]^T \end{array} \right\} \tag{3.4}$$

$$\left\{\hat{\$}^a_{ef}\right\} = \left\{ \begin{array}{c} \left[\begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 0 \end{array} \right]^T \\ \left[\begin{array}{cccccc} 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right]^T \\ \left[\begin{array}{cccccc} 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right]^T \\ \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 1 & -\frac{e_z}{e_y} \end{array} \right]^T \end{array} \right\} \tag{3.5}$$

Equation (3.4) represents two moments along x-axis and z-axis. On the other hand, Equation (3.5) represents two moments along y-axis and z-axis and two forces along the x-axis and the axis of link 5.

Thus the common constraint of limbs $a-1-b-2-c-3-d$ and $f-5-e$ can be written as:

$$\left\{\hat{\$}^a_{abcd}\right\} \cap \left\{\hat{\$}^a_{ef}\right\} = \left[\begin{array}{cccccc} 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right]^T \tag{3.6}$$

which is exactly the constraint of rotation around z-axis, as expected.

By a closer inspection of mechanism joints it can be observed that joints a, b, c, d, e and f are all capable of transmitting a torque around axis z. Applying screw theory, the same can be stated as: a circuit action, *i.e.* an action screw $\$^a$, is reciprocal to all motion screws associated with mechanism joints:

$$\{\$^a | \$^a \circ \$^m = 0,\ \forall \$^m \in \{a, b, c, d, e, f\}\} \tag{3.7}$$

where the reciprocity condition of a pair of screws has been defined as the scalar product in Equation (2.5).

In general, considering a single-loop kinematic chain, the two following statements are equivalent:

1. Two or more sets constituted by different links and joints constraint the same degree of freedom, *i.e.* a redundant constraint is present.

2. All joints can transmit the same action.

Considering the first statement, the joints of each set must be capable of transmitting the same action in order to constrain the same degree of freedom in the mechanism, it follows that all joints can transmit the same action. Regarding the second statement, consider two arbitrary chosen links of the mechanism and the two sets of links and joints connecting those links. As all joints can transmit the same action, it follows that the two sets constrain the same degree of freedom in the mechanism. The same result can be easily extended to multi-loop kinematic chain.

In Figure 9 the circuit action is represented as a screw with infinity pitch (a moment) along z-axis. A circuit action acting on motion screws of a kinematic chain performs no work when relative motion takes place in the joints. As a consequence, a strain energy can be locked inside the loop, which cannot be dissipated by link movements.

For a better understanding of how a generalised force can be locked inside a loop, the same mechanism of Figure 9 is considered with the loop open at joint d, as depicted in Figure 10.

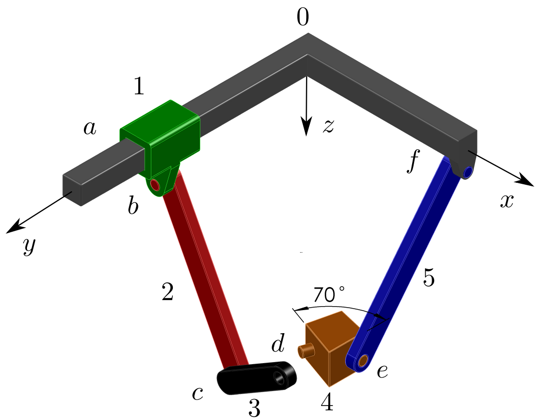


Figure 10 – Mechanism of Figure 9 with misalignment

In this mechanism, link 4 has been manufactured erroneously, with the two faces containing revolute kinematic pairs at an angle of 70° (instead of 90°). Because of this manufacturing error, a force/torque must be applied in order to deform the links and close the loop.

In Figure 11 the assembled mechanism is presented, with the loop closed. In order to simplify visualization, the whole deformation and consequent strain energy is attributed to link 0, which is depicted as flexible link formed by two rigid bodies connected by a revolute coupling along z-axis and a torsional spring. In the unstressed configuration, the two bodies composing link 0 are at right angle, while after closing the loop, they are forced at 110° and the deformation energy can be regarded as elastic potential energy stored in the torsional spring.

For a better understanding the concept of circuit action, a further

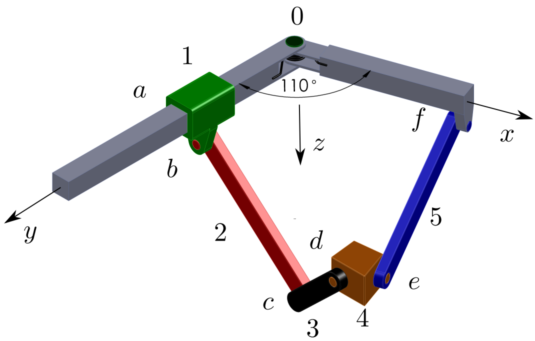


Figure 11 – Mechanism of Figure 9 with deformation

example, introduced by Davies(37), is herein presented. Consider a single loop obtained by bending a strip of steel (heated up until it becomes ductile) and welding the two edges together. Locked into the ring is a circuit action in the form:

$$\hat{\$}^a = \begin{bmatrix} R \\ S \\ T \\ U \\ V \\ W \end{bmatrix} \tag{3.8}$$

with respect to coordinate system. Two flanges of a hinge are now bolted to the strip now formed into a ring, with the axis of the hinge lying on the z-axis of the coordinate system. Next, the original ring is carefully cut between the flanges of the hinge. The ring remains a single integral body but is now hinged to itself. A small rotation is expected to take place in the ring around z-axis. At the same time the circuit action is affected by the cut because the hinge, which must now transmit the circuit action, is incapable of transmitting torque T parallel to the z-axis. Thus the circuit action has now the general form:

$$\hat{\$}^a = \begin{bmatrix} R \\ S \\ 0 \\ U \\ V \\ W \end{bmatrix} \tag{3.9}$$

On the other hand, the only motion allowed by the couplings of the loop, *i.e.* by the hinge, is a rotation along the z-axis in the form:

$$\hat{\$}^a = \begin{bmatrix} 0 \\ 0 \\ t \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.10}$$

It can now be verified that the circuit action of Equation 3.9 cannot expend power on the motion screw of Equation 3.10, as applying Equation (2.4):

$$(0, 0, t, 0, 0, 0) \cdot (R, S, 0, U, V, Z) = 0 \tag{3.11}$$

In the same way, a second hinge along x-axis can be added to the ring, reducing the order of the circuit action.

In this way, circuit actions are associated with redundant constraints. Moreover, the number of redundant constraints of a kinematic chain, *i.e.* the degree of redundant constraint of a kinematic chain, may be regarded as the number of free components of the circuit actions (37), introduced in Section 2.2.1. Considering a loop of a mechanism, the maximum number of redundant constraints in the circuit, *i.e.* the maximum number of free components of the circuit actions in the circuit, is six. Intuitively, this result can be explained in the following way: two links 1 and 2 in the loop are considered and the loop is opened at the two links. The two limbs joining the links are examined separately. Each limb imposes a maximum of six constraints between links 1 and 2. Moreover, no redundant constraints are present in the limbs, as the limbs are serial chains.

When the closed circuit is considered, the limbs can constraint at maximum the same six degrees of freedom, thus six redundant constraints are present in the loop. It suffices to notice that, when a limb imposes six constraints between two links, no motion is left between the links, thus they can be regarded as a single body. Therefore, when six redundant constraints are present in a loop, the entire loop collapse into a single body. This result will be discussed in further details in Section 3.2.

Just as the degree of freedom of a kinematic chain is the number of independent variables necessary to define all actions, the degree of redundancy C_N of a kinematic chain is the number of variables necessary to define all generalised forces that could be locked in the network as a consequence of the closure of circuits.

Also, the degree of redundancy of a single-loop mechanism is the order of the screw system reciprocal to the characteristic screws of the joints, *i.e.* to the screws describing the motions allowed by the joints. Waldron(38) correlates the order of this reciprocal screw system to the accuracy which the links of a mechanism must be manufactured.

3.2 NETWORK UNIT MOTION MATRIX ANALYSIS

In this section, the analysis of network unit motion matrix $\hat{\boldsymbol{M}}_N$ (17) is introduced. In Davies' papers (17, 4) matrix $\hat{\boldsymbol{M}}_N$ analysis is mostly focused on determining motion screw magnitudes in terms of a set of free variables. In (65, 37, 17), Davies also introduces a set of equations for determining the degree of overconstraint of a mechanism, as presented in the previous chapter. In this section, the analysis of the degree of overconstraint by means of matrix $\hat{\boldsymbol{M}}_N$ is extended introducing a set of linear algebra tools. As a result, the components of redundant constraints can be easy visualised for each loop of the mechanism.

A closer look insight the equations is presented, as this analysis will be further developed in the next sections. Results from electrical network theory are introduced and then applied to mechanical networks.

Given any electrical network, represented by a graph with g edges and n nodes, the principle of energy conservation applies (66):

$$\left[\boldsymbol{v}^{\boldsymbol{T}}\right]_{1,g} [\boldsymbol{i}]_{g,1} = 0 \tag{3.12}$$

where the component of vector $\left[\boldsymbol{v}^{\boldsymbol{T}}\right]_{1,g}$ are the instantaneous values of the potential differences across the network elements, represented by edge in the network graph, and the $[\boldsymbol{i}]_{g,1}$ are the instantaneous values of the current flowing through the same elements. Equation (3.12) is known as Tellegen's Theorem (67).

Considering that for any network the number of edges g exceeds the number of fundamental circuits ν, the information about current flowing is provided more concisely by the circuit current i_c (66), where the relation between $[\boldsymbol{i}]_{g,1}$ and $[\boldsymbol{i_c}]_{\nu,1}$ is:

$$[\boldsymbol{i}]_{g,1} = \left[\boldsymbol{B}^{\boldsymbol{T}}\right]_{g,\nu} [\boldsymbol{i_c}]_{\nu,1} \tag{3.13}$$

where $[\boldsymbol{B}]_{\nu,g}$ is the circuit matrix of network graph.

Thus Equation (3.12) can be expressed as:

$$\left[\boldsymbol{v^T}\right]_{1,g}\left[\boldsymbol{B^T}\right]_{g,\nu}\left[\boldsymbol{i_c}\right]_{\nu,1} = 0 \tag{3.14}$$

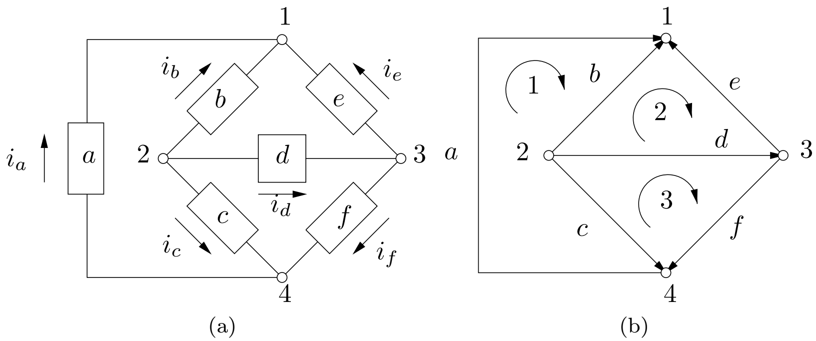


Figure 12 – Electrical circuit (a) and its graph representation (b)

An example is presented for a better understanding. In Figure 12a an electric circuit is depicted. The boxes $a, b, \ldots, f$ represent resistances or electric sources indistinctly.

The currents i can be written in terms of the circuit currents i_c:

$$\begin{aligned}
i_a(t) =& i_{C_1}(t) \\
i_b(t) =& -i_{C_1}(t) + i_{C_2}(t) \\
i_c(t) =& i_{C_1}(t) - i_{C_3}(t) \\
i_d(t) =& -i_{C_2}(t) + i_{C_3}(t) \\
i_e(t) =& -i_{C_2}(t) \\
i_f(t) =& i_{C_3}(t)
\end{aligned} \tag{3.15}$$

or more concisely:

$$\boldsymbol{i}(t) = [\boldsymbol{B}]^T \boldsymbol{i_c}(t) \tag{3.16}$$

where the circuit matrix $\boldsymbol{B}$ is:

$$[\boldsymbol{B}] = \begin{array}{c c} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{array}{c} C_1 \\ C_2 \\ C_3 \end{array} & \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 \end{bmatrix} \end{array} \tag{3.17}$$

and $\boldsymbol{i_c}(t) = \begin{bmatrix} i_{C_1} & i_{C_2} & i_{C_3} \end{bmatrix}^T$.

In mechanical networks an electrical-mechanical analogy is employed: an action screw is associated with current and motion screw with voltage. Circuit actions in mechanical networks are analogous to circuit currents: they provide the same information as coupling actions, but in a more concise form. A circuit action cannot expend power on any of the motions that the couplings in the circuit allow when they are unconstrained by circuit closure (17).

An example of mechanical network, *i.e.* a mechanism, is herein presented for better introducing the next steps of analysis. Consider the spatial four-bar mechanism in Figure 13, with four revolute couplings a, b, c and d around z-axis, and all links lying on the same plane.

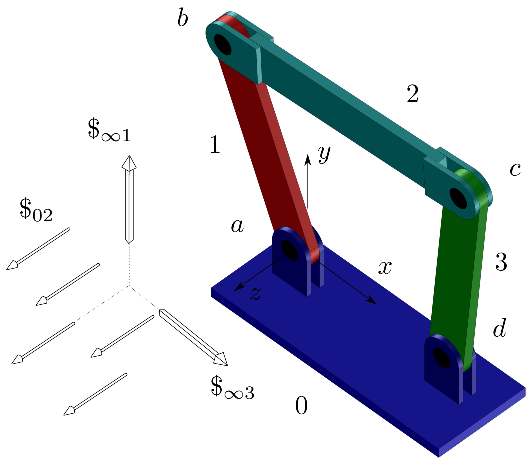


Figure 13 – Spatial four-bar mechanism

Figure 14 shows the structural representation of the four-bar mechanism, with link dimensions and joint positions. A coupling i in a mechanical network allows, in general, f_i freedoms. Each one of the couplings a, b, c and d allows one degree of freedom, thus the gross degree of freedom of the mechanism, *i.e.* the sum of the dof f_i of the couplings, is $F = 4$.

In general, for a given a mechanical network represented by a graph G_C, a new graph G_M with F edges, called motion graph, is obtained as described in Section 2.2.1. For the mechanism considered,

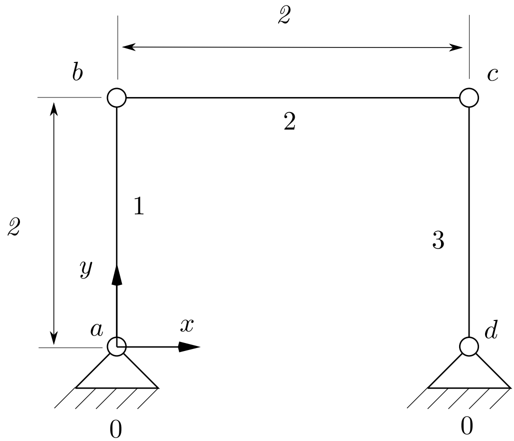


Figure 14 – Structural representation of mechanism in Figure 13

because joints a, b, c and d present each one single coupling freedom, graphs G_C and G_M are coincident. The graph G_C of the mechanism is represented in Figure 15.

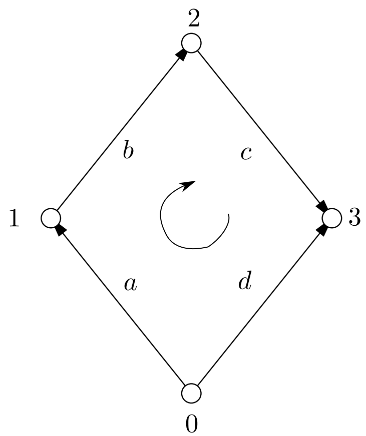


Figure 15 – Graph representation of mechanism in Figure 13

In the most general case, with the order of the screw system to which all screws under consideration belong $\lambda = 6$, a circuit action can be generally represented by an action screw in axis-coordinates:

$$\boldsymbol{\$^a} = \begin{bmatrix} R \\ S \\ T \\ U \\ V \\ W \end{bmatrix}_{6,1} \tag{3.18}$$

where the 6 are unknown.

Considering a single loop mechanism, the circuit actions are reciprocal to the F coupling motions spanning the f-systems of all the circuit couplings. Thus, F reciprocity equations can be written expressing conditions that the λ unknown circuit action components must satisfy. Moreover, the circuit action must be incapable of doing work on the displacement of any joint, which means that must be an action screw reciprocal to the motion screws of every joint of the circuit. This condition is expressed as:

$$\left[\boldsymbol{\hat{M}_D^T}\right]_{F,6} \begin{bmatrix} R \\ S \\ T \\ U \\ V \\ W \end{bmatrix}_{6,1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{F,1} \tag{3.19}$$

where matrix $\left[\boldsymbol{\hat{M}_D}\right]_{6,F}$ is the unit motion matrix, whose columns are the motion screws of the couplings. Equation (3.19) states that a circuit action cannot expend power on any of the motions allowed by the couplings of the circuit.

For the four-bar mechanism analysed in this particular configuration, the unit motion matrix is:

$$\left[\boldsymbol{\hat{M}_D}\right]_{6,4} = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \tag{3.20}$$

For the kinematic chain comprising a single circuit of $F = 4$ coupling motions and $n = 4$ links, equation (3.19) imposes F conditions to be met by the $\lambda = 6$ components of the circuit action. Equation (3.19) can be regarded as an homogeneous linear system with F equations and $\lambda = 6$ unknowns. Equation (3.19), for the mechanism of Figure 13 can be written as:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \end{bmatrix}_{4,6} \begin{bmatrix} R \\ S \\ T \\ U \\ V \\ W \end{bmatrix}_{6,1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{4,1} \tag{3.21}$$

It suffices to remember from linear algebra that an homogeneous system is always consistent (68). If the rank of matrix $\left[\hat{\boldsymbol{M}}_{\boldsymbol{D}}^{\boldsymbol{T}}\right]_{F,\lambda}$ is λ, than λ equations can be solved by Cramer's rule for the unique solution $A_{11} = A_{12} = \ldots = A_{1\nu} = 0$ and the system has only the trivial solution. In this case no redundant constraint is present in the mechanism.

If the rank of $\left[\hat{\boldsymbol{M}}_{\boldsymbol{D}}^{\boldsymbol{T}}\right]_{F,\lambda}$ is $r < \lambda$ then, the following theorem can be applied

Theorem 1. *(68) In a consistent system with λ unknowns of rank $r < \lambda$, $\lambda - r$ of the unknowns may be chosen so that the coefficient matrix of the remaining r unknowns is of rank r. When these $\lambda - r$ unknowns are assigned any whatever values, the other r unknowns are uniquely determined.*

Than by Theorem 1 the following statements are true:

- A necessary and sufficient condition for the homogeneous system (3.21) to have a solution other than the trivial solution is that the rank of $\left[\hat{\boldsymbol{M}}_{\boldsymbol{D}}^{\boldsymbol{T}}\right]_{F,\lambda}$ be $r < \lambda$. If $r = \lambda$, the only solution is the trivial one.
- If the rank of $\left[\hat{\boldsymbol{M}}_{\boldsymbol{D}}^{\boldsymbol{T}}\right]_{F,\lambda}$ is $r < \lambda$, the system has exactly $\lambda - r$ linearly independent solutions such that every solution is a linear combination of these $\lambda - r$ and every such linear combination is a solution.

The rank of $\left[\hat{\boldsymbol{M}}_{\boldsymbol{D}}^{\boldsymbol{T}}\right]_{6,4}$ in Equation (3.21) is $r = 3$, thus applying Theorem 1 Equation (3.21) states that the mechanism of Figure 13 has

$C_N = 6 \cdot 1 - 3 = 3$ redundant constraints. This result is expressed by Davies (17) in the form:

$$C_N = F - rank(\left[\boldsymbol{\hat{M}_D^T}\right]_{F,\lambda}) \tag{3.22}$$

or for multi-loop mechanical network as:

$$C_N = F - rank(\left[\boldsymbol{\hat{M}_N^T}\right]_{F,\lambda\nu}) \tag{3.23}$$

By elementary row transformations Equation (3.21) can be rearranged as:

$$\begin{bmatrix} 0 & 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{4,6} \begin{bmatrix} R \\ S \\ \boxed{T} \\ \boxed{U} \\ \boxed{V} \\ W \end{bmatrix}_{6,1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{4,1} \tag{3.24}$$

Matrix $\left[\boldsymbol{\hat{M}_D^T}\right]_{4,6}$ has been arranged in reduced row echelon form $(rref)$. Thus it has columns 3, 4 and 5 as pivots columns, and columns 1, 2 and 6 as free columns. Accordingly R, S and W are the free variables of the circuit action generated by the $C_N = 3$ redundant constraints:

$$\boldsymbol{\hat{\$}^a} = \begin{bmatrix} R & S & 0 & 0 & 0 & W \end{bmatrix}^T \tag{3.25}$$

The action screw associated with the circuit can be a force with a line of action along any line parallel with the z-axis, as it is expected for a planar mechanism.

The motion screws of the coupling network shown in Figure 13 belong to the fifth special three-system of screws (2). The action screws of Equation (3.25) also belong to the fifth special three-system of screws. The screw system describing the overconstraint of the mechanism represents the general constraint of the planar motion, as expected. The circuit action screws of Equation (3.25) are represented in Figure 13, with the notation introduced in Section 2.1.

Based on this analysis, mechanism of Figure 13 can be modified in order to eliminate some or all redundant constraints depicted. By Equation (3.25), a torque around x-axis, a torque around y-axis and a force along z-axis can be locked in single loop. In order to eliminate the $C_N = 3$ redundant constraints, angular mobility around x and y axes and translation mobility along z must be introduced in the joints a, b, c and d. In this way, torques R and S and force W can be dissipated within joint displacements and are no longer reciprocal to the motion screws of all joints of the loop.

This result can be achieved in different ways. In Figure 16 a new mechanism is presented, derived from the one in Figure 13. The redundant constraints of Equation (3.25) have been eliminated by introducing additional coupling motions. Two angular mobilities, respectively around x and y axes, have been added to joint b, which can be thus represented as a spherical joint. A translational mobility along z-axis has been introduced in joint d, which can be thus represented as a cylindrical one.

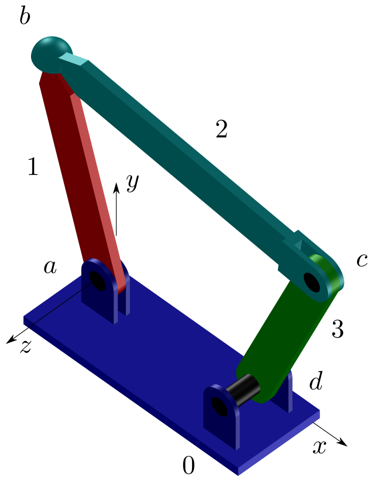


Figure 16 – Self-aligning mechanism derived from Figure 13, $F_N = 1$ and $C_N = 0$.

Mechanisms of Figures 13 and 16 have both mobility $F_N = 1$ and are kinematically equivalent, *i.e.* they have the same topology, the same mobility, the same number and dimension of links and the same number of circuits. Moreover, the links of the two mechanisms have the same motions.

The coupling mobilities added to the joints b and d do not affect the mobility of the mechanism, they only eliminate the redundant constraints present in the first mechanism.

An alternative mechanism can be derived from Figure 13, as presented in Figure 17.

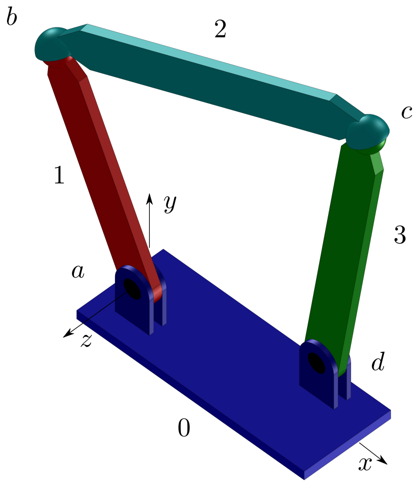


Figure 17 – Alternative self-aligning mechanism derived from the mechanism in Figure 13.

In this mechanism, R joints b and c have been transformed into spherical joints, introducing four supplementary coupling mobilities. The $C_N = 3$ redundant constraints present in the original mechanism have been thus eliminated, and an additional mobility is left in the mechanism. Link 2 is now capable of rotating along its own axis. The main mobility of the mechanism is unchanged, while the new mobility introduced is called *passive mobility* (46).

Recalling Equation 3.19 and Theorem 1, it can be verified that the number of free components, *i.e.* the number of redundant constraints, is $C_N = \lambda - r$ where r is the rank of matrix $\left[\hat{\boldsymbol{M}}_{\boldsymbol{D}}^{\boldsymbol{T}}\right]$. Therefore, as $\lambda \leq 6$, the maximum number of redundant constraints is $C_N = 5$ when the rank of $\left[\hat{\boldsymbol{M}}_{\boldsymbol{D}}^{\boldsymbol{T}}\right]$ is $r = 1$. If $r = 0$ and $C_N = 6$, it implies that $\left[\hat{\boldsymbol{M}}_{\boldsymbol{D}}^{\boldsymbol{T}}\right]$ is a zero matrix, or alternatively, that no freedom is allowed by the joints of the mechanism. In this case the entire loop collapses into a single body, as already introduced in the previous section.

3.2.1 Multi-loop mechanism analysis

A mechanical network with n links, g joints allowing F coupling motions, ν independent circuits is now considered, with λ being the minimum order of the screw system to which both motion and action screws under consideration belong to.

Each circuit action can be represented as an action screw $\hat{\$}^{\boldsymbol{a}}$ with λ unknown components, as in Equation (3.18). Thus the circuit action screws $\hat{\$}^{\boldsymbol{a}}_{\boldsymbol{1}}$, $\hat{\$}^{\boldsymbol{a}}_{\boldsymbol{2}}$, $\cdots$, $\hat{\$}^{\boldsymbol{a}}_{\boldsymbol{\nu}}$ of the mechanical network present a total of $\lambda\nu$ unknown components. As in the single-loop mechanical network, each coupling motion imposes a condition above the $\lambda\nu$ components of the circuit actions. More precisely, considering the motion graph G_M, if an edge belongs to only one independent circuit, the motion screws of the joint is reciprocal to the circuit action screw. If an edge belongs to more than one independent circuit, the characteristic screws of the joint is reciprocal to the resultant of the circuit screws of the loops to which the edge belongs.

These F conditions can be conveniently written in matricial form:

$$\begin{bmatrix} b_{11}\left[\hat{\$}^{\boldsymbol{m}}_{\boldsymbol{1}}\right]^T & b_{21}\left[\hat{\$}^{\boldsymbol{m}}_{\boldsymbol{1}}\right]^T & \cdots & b_{\nu 1}\left[\hat{\$}^{\boldsymbol{m}}_{\boldsymbol{1}}\right]^T \\ b_{12}\left[\hat{\$}^{\boldsymbol{m}}_{\boldsymbol{2}}\right]^T & b_{22}\left[\hat{\$}^{\boldsymbol{m}}_{\boldsymbol{2}}\right]^T & \cdots & b_{\nu 2}\left[\hat{\$}^{\boldsymbol{m}}_{\boldsymbol{2}}\right]^T \\ & \vdots & & \\ b_{1F}\left[\hat{\$}^{\boldsymbol{m}}_{\boldsymbol{F}}\right]^T & b_{2F}\left[\hat{\$}^{\boldsymbol{m}}_{\boldsymbol{F}}\right]^T & \cdots & b_{\nu F}\left[\hat{\$}^{\boldsymbol{m}}_{\boldsymbol{F}}\right]^T \end{bmatrix}_{F,\lambda\nu} \begin{bmatrix} \$^{\boldsymbol{a}}_{\boldsymbol{1}} \\ \$^{\boldsymbol{a}}_{\boldsymbol{2}} \\ \vdots \\ \$^{\boldsymbol{a}}_{\boldsymbol{\nu}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{\lambda\nu} \tag{3.26}$$

where b_{ij} (with $i = 1, 2, \dots, \nu$ and $j = 1, 2, \dots, F$) are the elements of the circuit matrix $[\boldsymbol{B_M}]_{\nu,F}$ of the motion graph G_M, $\$^m_k$ (with $k = 1, 2, \dots, F$) are the independent motion screws of each coupling and $\$^{\boldsymbol{a}}_{\boldsymbol{t}}$ (with $t = 1, 2, \dots, \nu$) are the circuit action screws of each fundamental circuit. Equation (3.26) can be concisely written (37) as:

$$\left[\hat{\boldsymbol{M}}^{\boldsymbol{T}}_{\boldsymbol{N}}\right]_{F,\lambda\nu} [\boldsymbol{A_l}]_{\lambda\nu,1} = [\boldsymbol{0}]_{F,1} \tag{3.27}$$

where the column vector $[\boldsymbol{A_l}]_{\lambda\nu,1}$ contains the $\lambda\nu$ components of the circuit actions:

$$[\boldsymbol{A_l}]_{\lambda\nu,1} = \begin{bmatrix} \left.\begin{matrix} A_{11} \\ \vdots \\ A_{1\lambda} \end{matrix}\right\} \text{Circuit 1} \\ \left.A_{21}\right\} \text{Circuit 2} \\ \vdots \\ \left.A_{\nu\lambda}\right\} \text{Circuit } \nu \end{bmatrix}_{\lambda\nu,1} \tag{3.28}$$

and matrix $\left[\boldsymbol{\hat{M}_N^T}\right]_{F,\lambda\nu}$, also called freedom matrix (17), imposes F conditions on the $\lambda\nu$ components of the circuit actions. Equation (3.27) is an homogeneous linear system of F equations in the $\lambda\nu$ unknowns. In Section 3.2.1.1 a numerical example is presented.

Equation (3.27) imposes F conditions on the $\lambda\nu$ unknown components of the circuits action. Thus Equation (3.27) reduces the $\lambda\nu$ unknown components by r, the rank of the freedom matrix $\left[\boldsymbol{\hat{M}_N^T}\right]_{\lambda\nu,F}$.

The $\lambda\nu - r$ are therefore free variables, and the number C_N of redundant constraint in a given kinematic chain is therefore obtained as (17):

$$C_N = \lambda\nu - r \tag{3.29}$$

C_N can be regarded as the degree of redundant constraint of the mechanism.

Redundant constraints can be now further analysed, by applying linear algebra tools. In Appendix A main definitions and theorems are presented.

The freedom matrix $\left[\boldsymbol{\hat{M}_N^T}\right]_{\lambda\nu,F}$ can be conveniently expressed in reduced row echelon form (*rref*). Equation (3.27) is therefore:

$$
\begin{bmatrix}
\boxed{1} & \ast & \ast & 0 & \ast & 0 & \ast & \ast & \\
 & \ast & \ast & 0 & \ast & 0 & \ast & \ast & \\
0 & & & \ddots & 0 & & 0 & & \\
 & & & \boxed{1} & \ast & 0 & \ast & \ast & \\
\vdots & & & & & \ddots & 0 & \ast & \ast \\
 & & & & & \boxed{1} & \ast & \ast & \\
0 & & & & & & & 0 &
\end{bmatrix}_{F,\lambda\nu}
\begin{bmatrix}
A_{11} \\
\boxed{A_{12}} \\
\vdots \\
A_{1\lambda} \\
\boxed{A_{21}} \\
\vdots \\
\boxed{A_{ij}} \\
\vdots \\
A_{\nu\lambda}
\end{bmatrix}_{\lambda\nu,1}
= [\mathbf{0}]_{F,1}
\tag{3.30}
$$

As described in Appendix A, when the coefficient matrix of a linear system is brought into *rref* form, the pivot columns, *i.e.* the columns containing the pivots 1, correspond to the leading variables, while the remaining variables are free variables.

In Equation (3.30) the squares contain the pivot values. The r columns of pivot values correspond to the r pivot variables, also called leading variables or primary variables, represented inside a square. The other $\lambda\nu - r$ variables are free variables, which can assume arbitrary values.

Thus, given a mechanism represented by a mechanical coupling network by means of Equation (3.30), the component of redundant constraints for each circuit are detected and circuit actions can be described in terms of the free variables.

This result is of great importance for the analysis of an existing mechanism or the design of a new one. For each loop, a description of the circuit actions in terms of components is given. If overconstraint is present in some loops of the mechanism, the free variable of circuit actions can be totally or partially eliminated by proper increase in joints' mobility. In this way redundant constraints are eliminated, turning the mechanism in a self-aligning one. Alternatively, when overconstraint is necessary, for rigidity increase or better loads distributions in couplings, strict tolerances can be required in links manufacturing. In this way a mechanism can be composed of self-aligning loops and overconstrained ones, in order to better fit the required design specifications. Further analysis tools are provided in this Thesis, which eventually result in a complete method to deal with overconstraint in mechanisms. Some examples of application are presented in the following sections.

3.2.1.1 Example I: multi-loop planar kinematic chain

The analysis of redundant constraint with the method presented in the previous section can be performed on complex coupling mechanisms. Consider the planar mechanism of Figure 18.

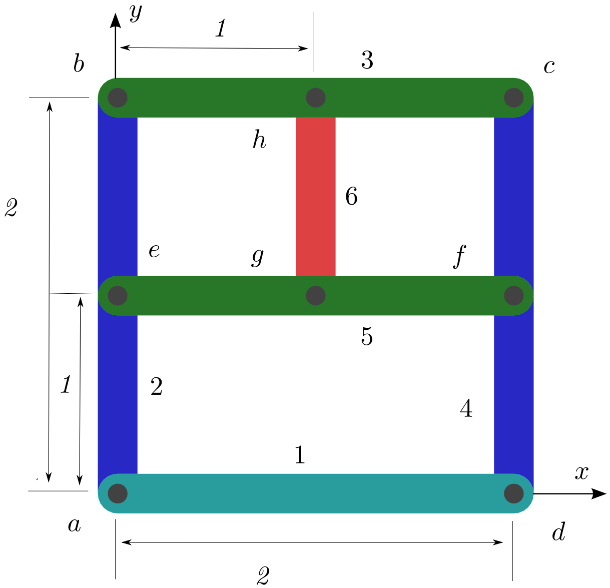


Figure 18 – Planar mechanism with $n = 6$ links and $g = 8$, $F_N = 1$ and $C_N = 2$.

The graph of this mechanism is presented in Figure 19, with the $\nu = 3$ fundamental circuits. All couplings a, b, c, d, e, f, g, h allow one degree of freedom each, thus the coupling graph G_C and the motion graph G_M are coincident for the mechanism considered.

As the mechanism is planar, only the components of planar motion are considered, *i.e.* t, u and v. The motion screws of the coupling network thus belong to the fifth special three-system of screws (2), *i.e.* two linear velocities u and v ($h \to \infty$) and one angular velocity t ($h = 0$). The circuit action screws caused by overconstraint must therefore belong to the fourth special 3-system of screws, or a subsystem of it. The mechanism considered is planar, thus the focus is on

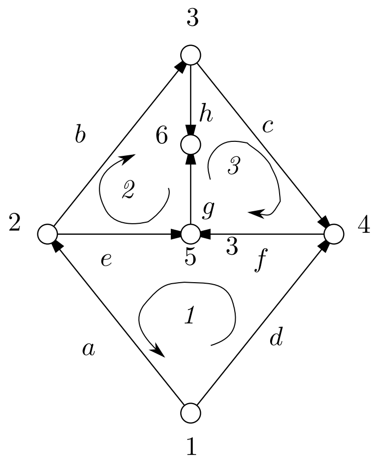


Figure 19 – Coupling graph G_C and motion graph G_M of mechanism in Figure 18.

the planar redundant constraints. In this case the spatial constraints, *i.e.* R, S and W, are ignored. Therefore the planar circuit actions belong to the set T, U, V, *i.e.* two forces U and $V(h = 0)$ and one moment T $(h \to \infty)$.

Based on the fundamental circuits identified in the graph G_M of the mechanism, presented in Figure 19 ,the circuit matrix $[\boldsymbol{B_M}]$ can be written as:

$$[\boldsymbol{B_M}]_{3,8} = \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \left[\begin{array}{cccccccc} -1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & -1 \end{array}\right] \end{array} \tag{3.31}$$

Thus the unit motion matrix $\boldsymbol{\hat{M}_D}$ can be written as:

$$[\boldsymbol{\hat{M}_D}]_{3,8} = \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \left[\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & -2 & -2 & 0 & -2 & -1 & -1 \end{array}\right] \end{array} \tag{3.32}$$

Matrix $\boldsymbol{\hat{M}_N}$ can thus be obtained applying Equation (2.12). By means of Equation (3.27), the following homogenous linear system can now be written:

$$\begin{bmatrix} -1 & 0 & 0 & | & 0 & 0 & 0 & | & 0 & 0 & 0 \\ 0 & 0 & 0 & | & 1 & 2 & 0 & | & 0 & 0 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 0 & | & 1 & 2 & -2 \\ 1 & 0 & -2 & | & 0 & 0 & 0 & | & 0 & 0 & 0 \\ -1 & -1 & 0 & | & -1 & -1 & 0 & | & 0 & 0 & 0 \\ 1 & 1 & -2 & | & 0 & 0 & 0 & | & 1 & 1 & -2 \\ 0 & 0 & 0 & | & -1 & -1 & 1 & | & 1 & 1 & -1 \\ 0 & 0 & 0 & | & 1 & 2 & -1 & | & -1 & -2 & 1 \end{bmatrix}_{8,9} \begin{bmatrix} T_1 \\ U_1 \\ V_1 \\ -- \\ T_2 \\ U_2 \\ V_2 \\ -- \\ T_3 \\ U_3 \\ V_3 \end{bmatrix}_{9,1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{8,1} \tag{3.33}$$

The subscript in the unknown components of the circuit actions refers to the circuit to which the component belongs. As the mechanism is planar, only the planar components of circuit action are considered, *i.e.* T, U and V. Regard that the matrix on the left of Equation (3.33), *i.e.* $\hat{\boldsymbol{M}}_{\boldsymbol{N}}^T$, can be obtained alternatively by equation (3.26).

Equation (3.33) can be rearranged by means of elementary operations in the *rref* form as:

$$\begin{bmatrix} \boxed{1} & 0 & 0 & | & 0 & 0 & 0 & | & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & | & 0 & 0 & 0 & | & 0 & -1 & 0 \\ 0 & 0 & \boxed{1} & | & 0 & 0 & 0 & | & 0 & 0 & 0 \\ 0 & 0 & 0 & | & \boxed{1} & 0 & 0 & | & 0 & 2 & 0 \\ 0 & 0 & 0 & | & 0 & \boxed{1} & 0 & | & 0 & -1 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & \boxed{1} & | & 0 & 0 & 1 \\ 0 & 0 & 0 & | & 0 & 0 & 0 & | & \boxed{1} & 2 & -2 \\ 0 & 0 & 0 & | & 0 & 0 & 0 & | & 0 & 0 & 0 \end{bmatrix}_{8,9} \begin{bmatrix} \boxed{T_1} \\ \boxed{U_1} \\ \boxed{V_1} \\ -- \\ \boxed{T_2} \\ \boxed{U_2} \\ \boxed{V_2} \\ -- \\ \boxed{T_3} \\ U_3 \\ V_3 \end{bmatrix}_{9,1}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T_{1,8} \tag{3.34}$$

where matrix $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}^{\boldsymbol{T}}\right]$ is in reduced row echelon form. Columns $1-7$ are pivots columns and columns 8 and 9 are free columns. Therefore, U_3 and V_3 are the free components of circuit actions. The number of redundant constraints of the mechanism is $C_N = 2$. The circuit actions of the mechanism can be written in terms of the free components. For example considering the circuit 1 in Figure 19, two components of the circuit action are zero, more exactly $T_1 = 0$ and $V_1 = 0$ by means of Equation (3.34). On the other hand, the component U_1 can be written in terms of the free variables U_3, V_3 as $U_1 = U_3$. The circuit actions for the three circuits can be written as:

$$\begin{aligned} \hat{\$}_{\boldsymbol{1}}^{\boldsymbol{a}} &= \begin{bmatrix} 0 & U_3 & 0 \end{bmatrix}^T \\ \hat{\$}_{\boldsymbol{2}}^{\boldsymbol{a}} &= \begin{bmatrix} -2U_3 & U_3 & -V_3 \end{bmatrix}^T \\ \hat{\$}_{\boldsymbol{3}}^{\boldsymbol{a}} &= \begin{bmatrix} -2U_3 + 2V_3 & U_3 & V_3 \end{bmatrix}^T \end{aligned} \tag{3.35}$$

where the subscript to the screw symbol indicates the circuit number. Equation (3.35) can be interpreted in the following way: two actions can be independently locked inside the mechanism of Figure 18. When these two actions, *i.e.* a force U_3 along x-axis and a force V_3 along y-axis (both in circuit 3), are present in the mechanism then the circuit actions described in Equation (3.35) arise in the circuits. The choice of free variables is not unique, thus different independent actions can be selected in term of which the circuit actions can be expressed.

As previously stated, the circuit action screws for a planar mechanism belong to the fourth special 3-system of screws, *i.e.* two forces U and V with $h = 0$ and one moment T with $h \to \infty$. For the mechanism considered, only the forces U and V are present, thus the circuit action screws belong to the first special two-system, which can be regarded as a subsystem of the fourth special three-system of screws, as expected.

The redundant constraints detected can be eliminated introducing additional coupling mobilities in the proper loops. As indicated by equation (3.35), forces along x and y-axes can be locked in loop 3. In order to eliminate the $C_N = 2$ redundant constraints, linear mobilities along x and y-axis must be introduced in the loop. The mechanism in Figure 20 is obtained increasing the mobilities of joint h and f, respectively with linear mobility along x and along y, as indicated by the double arrows.

The mechanism of Figure 20 is a self-aligning mechanism, kinematically equivalent to the mechanism presented in Figure 18. In fact, it can be verified that this new mechanism owns the same degree of free-

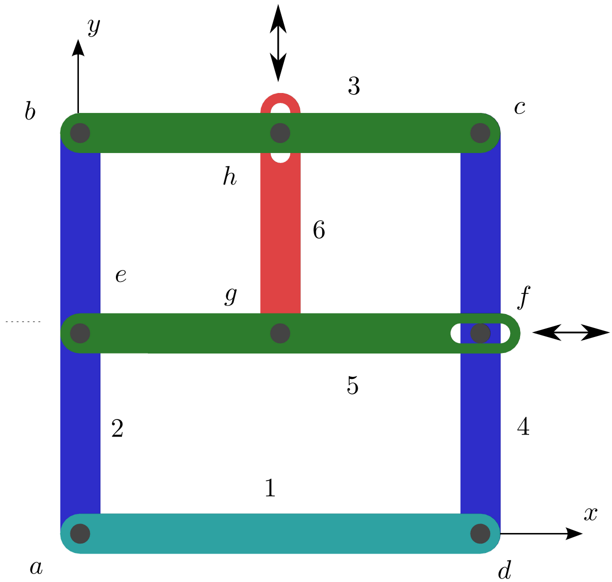


Figure 20 – Self-aligning mechanism derived from the one in Figure 18, $F_N = 1$ and $C_N = 0$.

dom, *i.e.* $F_N = 1$, the same links and presents the same motions of the mechanism in Figure 18. As no redundant constraints are present in the mechanism of Figure 20, this mechanism is self-aligning, thus no action can be locked inside the loop.

3.2.1.2 Example II: multi-loop spatial kinematic chain

Consider the spatial mechanism in Figure 21, the well know parallel mechanism *Tripteron* proposed by Kong and Gosselin(69) and built at Laval University (70).

The coupling graph G_C of this mechanism, coincident with the motion graph G_M, is represented in Figure 22.

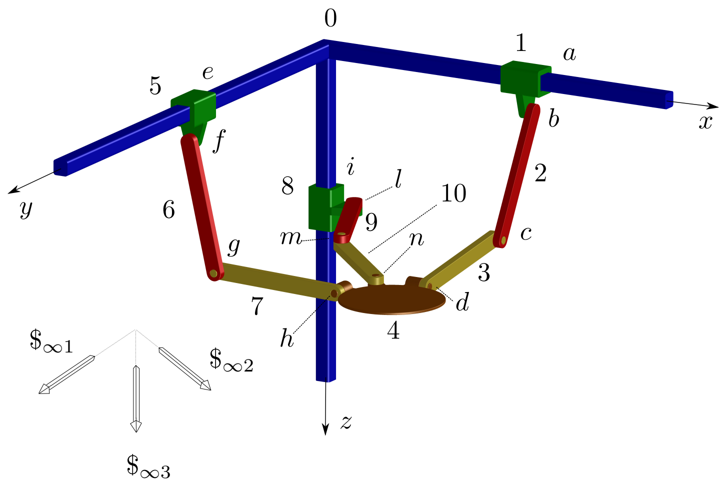


Figure 21 – Spatial mechanism Tripteron with $n = 11$ links and $g = 12$ joints, $F_N = 3$ and $C_N = 3$.

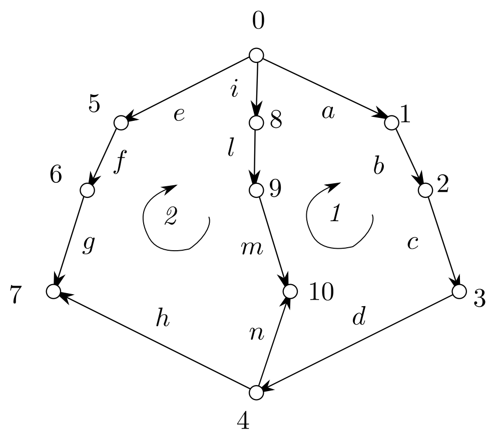


Figure 22 – Graph G_C and G_M of *Tripteron* mechanism

Thus the unit motion matrix $\hat{\boldsymbol{M}}_{\boldsymbol{D}}$ can be written as:

$$\hat{\boldsymbol{M}}_{\boldsymbol{D}3,8} = \begin{array}{c} \begin{array}{cccccccccccc} a & b & c & d & e & f & g & h & i & l & m & n \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -g_z & -h_z & 0 & 0 & m_y & n_y \\ 0 & 0 & -c_z & -d_z & 1 & 0 & 0 & 0 & 0 & 0 & -m_x & -n_x \\ 0 & 0 & c_y & d_y & 0 & 0 & g_x & h_x & 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \tag{3.36}$$

where i_x, i_y and i_z are the coordinate of joint i. Based on the graph G_M of the mechanism presented in Figure 22, the fundamental circuit matrix is:

$$\boldsymbol{B}_{2,11} = \begin{array}{c} \begin{array}{cccccccccccc} a & b & c & d & e & f & g & h & i & l & m & n \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 \end{bmatrix} \end{array} \tag{3.37}$$

By means of Equation (3.27), the following homogenous linear system can be written:

$$\left[\begin{array}{cccccc|cccccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & c_z & -c_y & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & d_z & -d_y & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & g_z & 0 & -g_x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & h_z & 0 & -h_x \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -m_y & m_x & 0 & 0 & 0 & 1 & m_y & -m_x & 0 \\ 0 & 0 & 1 & n_y & -n_x & 0 & 0 & 0 & -1 & -n_y & n_x & 0 \end{array}\right]_{12,12} \begin{bmatrix} R_1 \\ S_1 \\ T_1 \\ U_1 \\ V_1 \\ W_1 \\ -- \\ R_2 \\ S_2 \\ T_2 \\ U_2 \\ V_2 \\ W_2 \end{bmatrix}_{12,1}$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T_{1,12} \tag{3.38}$$

Equation (3.38) can be rearranged by means of elementary operations as:

$$\left[\begin{array}{cccccc|cccccc} \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]_{12,12} \left[\begin{array}{c} \boxed{R_1} \\ S_1 \\ \boxed{T_1} \\ \boxed{U_1} \\ \boxed{V_1} \\ \boxed{W_1} \\ -- \\ R_2 \\ \boxed{S_2} \\ T_2 \\ \boxed{U_2} \\ \boxed{V_2} \\ \boxed{W_2} \end{array}\right]_{12,1}$$

$$= \left[\begin{array}{cccccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]^{T}_{1,12} \tag{3.39}$$

Matrix $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}^{\boldsymbol{T}}\right]_{4,6}$ has been arranged in reduced row echelon form (*rref*). Thus it has columns 2, 7 and 9 as free columns, and the others columns are pivots. It suffices to notice that the pivot columns are linearly independent, while the free columns 2, 7 and 9 are linearly dependent, as introduced in Appendix A. Accordingly S_1, R_2 and T_2 are the free components of the circuit action generated by the $C_N = 3$ redundant constraints.

The number of redundant constraints of the mechanism is $C_N = 3$. The circuit actions of the mechanism can be written as:

$$\begin{aligned} \hat{\$}_{\boldsymbol{1}}^{\boldsymbol{a}} &= \left[\begin{array}{cccccc} 0 & S_1 & T_2 & 0 & 0 & 0 \end{array}\right]^T \\ \hat{\$}_{\boldsymbol{2}}^{\boldsymbol{a}} &= \left[\begin{array}{cccccc} R_2 & 0 & T_2 & 0 & 0 & 0 \end{array}\right]^T \end{aligned} \tag{3.40}$$

The circuit action screws of Equation (3.40) are represented in Figure 21, with the notation introduced in Section 2.1.

A new self-aligning mechanism can be derived from the *Tripteron* of Figure 21, eliminating the redundant constraint. Based on Equation (3.40), three additional angular mobilities must be added to the

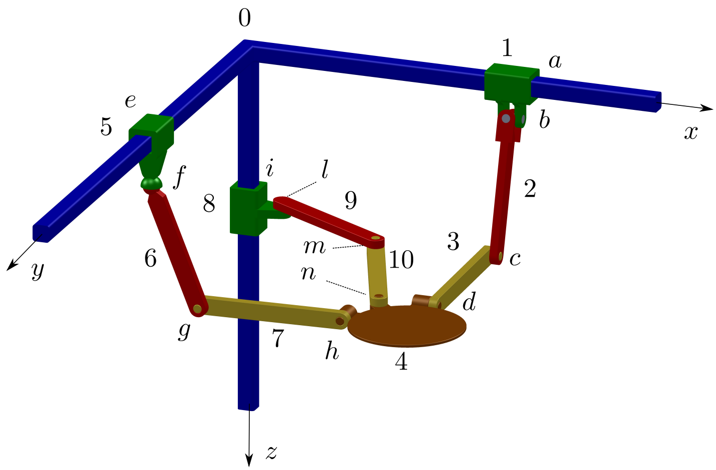


Figure 23 – Self-aligning Tripteron.

mechanism, more precisely an angular mobility around y-axis in loop 1, and two angular mobilities around x and z-axes in loop 2.

In Figure 23 two angular mobilities around x and z have been added to joint f, turning it into a spherical coupling. An additional angular mobility around y-axis has been added to joint b. Thus joint b, joining links 1 and 2, is a Hook joint with two degree of freedom $f_b = 2$, *i.e.* two angular mobilities around x and y axes. The new mechanism of Figure 23 is completely free of redundant constraints, and present the same mobility $F_N = 3$ as the original *Tripteron*. This mechanism can work without any additional strain even if some misalignment is present between the three prismatic guides composing link 0, when the mechanism is not in a singular configuration.

3.3 NETWORK UNIT ACTION MATRIX ANALYSIS

In this section, a network unit action matrix $\left[\hat{\boldsymbol{A}}_N\right]$ analysis is introduced. Davies (17, 16) employs the cutset law in order to determine the actions transmitted in a mechanism subjected to external loads. Thus the focus is on the overconstraint generated by the internalisation of external loads. In this Thesis a different approach is introduced. Only the inherent overconstraint of a given mechanism is considered.

Analysis of matrix $\left[\hat{\boldsymbol{A}}_N\right]$ is extended introducing a set of linear algebra tools. As a result, a new method for obtaining a self-aligning mechanism kinematically equivalent to a given one is presented.

Given a mechanical network represented by a coupling graph G_C with edges a, b,... a new graph G_A with C edges, called action graph, is considered, as introduced in Section 2.2.1 .

The cutset law equation, already referred presented as Equation (2.17), is herein repeated:

$$\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]_{\lambda k, C}[\boldsymbol{\Psi}]_C = [\mathbf{0}]_{\lambda k} \tag{3.41}$$

A closer look insight this equation is provided. The network unit action matrix is obtained as:

$$\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]_{\lambda k, C} = \begin{bmatrix} \left[\hat{\boldsymbol{A}}_{\boldsymbol{D}}\right]_{\lambda, C} & [\boldsymbol{Q_1}]_{C,C} \\ \left[\hat{\boldsymbol{A}}_{\boldsymbol{D}}\right]_{\lambda, C} & [\boldsymbol{Q_2}]_{C,C} \\ \vdots & \vdots \\ \left[\hat{\boldsymbol{A}}_{\boldsymbol{D}}\right]_{\lambda, C} & [\boldsymbol{Q_K}]_{C,C} \end{bmatrix} \tag{3.42}$$

where $[\boldsymbol{Q_i}]_{C,C}$, $i = 1, 2, \ldots, k$ are diagonal matrices whose diagonal elements corresponding to row i of the cutset matrix $[\boldsymbol{Q_A}]_{C,C}$. For a better understanding a numerical example is presented id Appendix C.1.

$\left[\hat{\boldsymbol{A}}_{\boldsymbol{D}}\right]$ is the unit action matrix containing one action screw for column:

$$\left[\hat{\boldsymbol{A}_{\boldsymbol{D}}}\right] = \left[\begin{array}{c:c:c:c} \overbrace{\$^a_{a1} \quad \$^a_{a2} \quad \ldots}^{\text{Coupling a}} & \overbrace{\$^a_{b1} \quad \ldots}^{\text{Coupling b}} & \overbrace{\ldots}^{\ldots} & \overbrace{\ldots \quad \$^a_C}^{\ldots} \end{array}\right] \tag{3.43}$$

where k is the number of the cutsets in the coupling network, λ is the order of the screw system to which all action screws belong (in the most general case $\lambda = 6$) and $C = \sum c_i$ is the gross degree of constraint, *i.e.* the sum of the degree of constraint c_i of the couplings of the kinematic chain.

The partitions lines separate the c_i independent actions of each coupling i, that together span the c-system of action screws characteristic of the coupling. The columns of matrix $\left[\hat{\boldsymbol{A}_{\boldsymbol{D}}}\right]_{\lambda, C}$ must be in the same sequence as the columns of $[\boldsymbol{Q_A}]_{C,C}$. Thus the network unit action matrix presents the following structure:

$$\left[\hat{\boldsymbol{A_N}}\right]_{\lambda k,C} = \begin{array}{c} \begin{array}{cccc} \overbrace{\qquad\qquad\quad}^{\text{Coupling a}} & \overbrace{\qquad\quad}^{\text{Coupling b}} & \overbrace{\qquad}^{\dots} & \overbrace{\qquad}^{\dots} \end{array} \\ \left[\begin{array}{ccc:cc:c:cc} * & * & \dots & * & \dots & \dots & \dots & * \\ * & * & \dots & * & \dots & \dots & \dots & * \\ & & & & & & & \\ * & * & \dots & * & \dots & \dots & \dots & * \\ & & & & & & & \\ * & * & \dots & * & \dots & \dots & \dots & * \\ * & * & \dots & * & \dots & \dots & \dots & * \\ & & & & & & & \\ * & * & \dots & * & \dots & \dots & \dots & * \\ & & & & & & & \\ * & * & \dots & * & \dots & \dots & \dots & * \\ * & * & \dots & * & \dots & \dots & \dots & * \\ & & & & & & & \\ * & * & \dots & * & \dots & \dots & \dots & * \end{array}\right] \end{array} \begin{array}{l} \left.\begin{array}{l} \}\ \text{dim. } 1 \\ \}\ \text{dim. } 2 \\ \qquad\vdots \\ \}\ \text{dim. } \lambda \end{array}\right\} k_1 \\ \left.\begin{array}{l} \}\ \text{dim. } 1 \\ \}\ \text{dim. } 2 \\ \qquad\vdots \\ \}\ \text{dim. } \lambda \end{array}\right\} k_2 \\ \qquad\qquad\quad\vdots \\ \left.\begin{array}{l} \}\ \text{dim. } 1 \\ \}\ \text{dim. } 2 \\ \qquad\vdots \\ \}\ \text{dim. } \lambda \end{array}\right\} k_k \end{array} \tag{3.44}$$

A different form for $\left[\hat{\boldsymbol{A_N}}\right]_{\lambda,C}$ can be computed (71). The form of Equation (3.44) will be adopted for the analysis presented in the following text.

Equation (3.41) is an homogeneous linear system, analogous to Equation (3.27). and it can be expanded as:

$$
\begin{array}{c}
\begin{array}{cccc}
\overbrace{\hphantom{* \quad * \quad \cdots}}^{\text{Coupling a}} & \overbrace{\hphantom{* \quad \cdots}}^{\text{Coupling b}} & \overbrace{\hphantom{\cdots}}^{\cdots} & \overbrace{\hphantom{\cdots \quad *}}^{\cdots}
\end{array}
\\
\left[\begin{array}{ccc:cc:c:cc}
* & * & \cdots & * & \cdots & \cdots & \cdots & * \\
* & * & \cdots & * & \cdots & \cdots & \cdots & * \\
 & & & & & & & \vdots \\
* & * & \cdots & * & \cdots & \cdots & \cdots & * \\
 & & & & & & & \\
* & * & \cdots & * & \cdots & \cdots & \cdots & * \\
* & * & \cdots & * & \cdots & \cdots & \cdots & * \\
 & & & & & & & \vdots \\
* & * & \cdots & * & \cdots & \cdots & \cdots & \\
* & * & \cdots & * & \cdots & \cdots & \cdots & * \\
* & * & \cdots & * & \cdots & \cdots & \cdots & * \\
 & & & & & & & \vdots \\
* & * & \cdots & * & \cdots & \cdots & \cdots & *
\end{array}\right]_{\lambda k, C}
\end{array}
\left[\begin{array}{c}
\Psi_{a1} \\ \Psi_{a2} \\ \vdots \\ -- \\ \Psi_{b1} \\ \vdots \\ -- \\ \cdots \\ -- \\ \vdots \\ \Psi_C
\end{array}\right]_{C,1}
= \left[\begin{array}{cccc} 0 & 0 & \ldots & 0 \end{array}\right]^T_{1,\lambda k}
\tag{3.45}
$$

where Ψ_{a1}, Ψ_{a2}, ..., Ψ_C are the unknown magnitudes of the action screws, in the same sequence as columns of matrices $\left[\hat{\boldsymbol{A}}_{\boldsymbol{D}}\right]_{\lambda,C}$ and $[\boldsymbol{Q}_{\boldsymbol{A}}]_{C,C}$.

It suffices to notice that each column of $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]_{\lambda k,C}$ is represents a single constraint (a single *doc*) belonging to a coupling, as indicated in Equation (3.45). Dependence and independence of the mechanism's constraints can thus be analysed by studying the properties of matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]_{\lambda k,C}$.

It is important to regard that equation (3.41) is an homogeneous linear system with λk equations in C unknowns. The C unknowns are the magnitudes of the actions screws representing the actions transmitted by the couplings of the mechanism.

Equation (3.41) has been used in statics by Davies(17) in order to determine the actions transmitted in a mechanism subjected to external loads. Essentially, this means that active couplings (such as engine, gravitation or any other source of power) are considered as passive couplings, by internalising external actions. The result coupling network is thus overconstrained, and a set of primary variables can be identified and the unknown magnitudes be expressed in terms of the primary ones.

The method proposed by Davies cannot distinguish between the actions attributable to the original overconstraint of the system and those that are internalised if, as usual, the coupling network is already overconstrained before the external actions are internalised. Therefore, relationships between all actions that could exist in the mechanism are provided, even if not all ones are required. Moreover, when overconstraint is present, constraint actions applied by the couplings are not solvable if constitutive equations of material are not considered.

Davies circumvents this difficulty by disregarding overconstraint in the mechanism considered. In (17) the example of a gear train is given, where the action space has been chosen as $\lambda = 2$, and all force are coplanar and parallel, in order to avoid any overconstraint inherent to the mechanism. Laus(71) proposes a a new method for action internalisation that is more straightforward than methods described by Davies, introducing a new kind of active coupling.

In this Thesis a new approach is introduced. Given a mechanism, active coupling, *i.e.* actuators, are disregarded, and all joints are considered as not actuated ones. Thus in Equation (3.41) only the actions attributable to the overconstraint inherent to the system are considered.

In a self-aligning mechanism, where no overconstraint is present, no action can be locked inside the mechanism. Furthermore, all actions attributable to overconstraint are identically null. Thus for a self-aligning mechanism, by Theorem (4), Equation (3.41) reduces to a homogeneous system with λk equations in C unknowns, and rank of the coefficient matrix equal to the number of unknowns. The only admissible solution is the trivial one, as expected. The same result is stated by Davies(17):

$$C_N = C - rank\left(\left[\hat{\boldsymbol{A_N}}\right]_{\lambda k, C}\right) = 0 \tag{3.46}$$

which states the degree of constraint for a self-aligning mechanism is 0.

When the given mechanism is overconstrained, a set of actions, the circuit actions, can be locked inside the loops. The magnitudes of the actions transmitted by couplings cannot be solved without considering the constitutive equations of the links, but these magnitudes can be expressed in terms of a set of C_N, *i.e.* the degree of redundant constraint of the mechanism, primary variables (17). It suffices to notice that the numerical values of the action magnitudes cannot be computed unless constitutive equations are considered, as any so-called hyper-static problem. Anyway, if a set of strain gages is affixed in order to measure the C_N primary selected actions, all other actions can found in terms of those ones.

Looking at the homogeneous system, an alternatively interpretation can be presented. Consider the column space of matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]_{\lambda k,C}$: nominally the nullspace $null(\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]_{\lambda k,C})$ constitutes the space of the constraints of the mechanism. When the columns of $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]_{\lambda k,C}$, form a basis for the column space, *i.e.* are linearly independent, no redundant constraint is present, and the mechanism is self-aligning. Otherwise, one or more columns are linearly dependent, which means that one or more constraints are redundant, and the mechanism is overconstrained.

In this way the problem of eliminating redundant constraints in a mechanism is transformed in the one of finding the dependent columns of matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]_{\lambda k,C}$. The constraints corresponding to the dependent columns can thus be eliminated in the mechanism by adding proper additional mobilities to the couplings. A new method, contribution of this Thesis, is introduced in the following step to achieve this result, based on the reduced row echelon form of $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]_{\lambda k,C}$. As a result, a self-aligning mechanism equivalent to the given one is derived.

3.3.1 RREF analysis of the network unit action matrix

Analogously to the analysis of the network unit motion matrix in Section 3.3, it is convenient to bring matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]_{\lambda k,C}$ in *rref.*

$$
\begin{array}{c}
\begin{array}{cccc}
\overbrace{\hphantom{\boxed{1}\;0\;\cdots}}^{\text{Coupling a}} & \overbrace{\hphantom{0\;\cdots}}^{\text{Coupling b}} & \overbrace{\hphantom{\cdots}}^{\cdots} & \overbrace{\hphantom{\cdots\;*}}^{\cdots}
\end{array}\\
\left[\begin{array}{ccc|cc|c|cc}
\boxed{1} & 0 & \cdots & 0 & \cdots & \cdots & \cdots & * \\
 & \boxed{1} & \cdots & 0 & \cdots & \cdots & \cdots & * \\
 & & & & & & & \vdots \\
 & & \cdots & \boxed{1} & \cdots & \cdots & \cdots & * \\
 & & & & & & & \\
 & & \cdots & & \cdots & \cdots & \cdots & * \\
 & & \cdots & & \cdots & \cdots & \cdots & * \\
 & & & & & & & \vdots \\
 & & \cdots & & \cdots & \cdots & \cdots & * \\
 & & \cdots & & \cdots & \cdots & \cdots & * \\
 & & \cdots & & \cdots & \cdots & \cdots & * \\
 & & & & & & & \vdots \\
 & & \cdots & & \cdots & \cdots & \cdots & *
\end{array}\right]_{\lambda k,C}
\end{array}
\left[\begin{array}{c}
\boxed{\psi_{a1}} \\
\boxed{\psi_{a2}} \\
\vdots \\
-- \\
\boxed{\psi_{b1}} \\
\vdots \\
-- \\
\cdots \\
-- \\
\vdots \\
\psi_C
\end{array}\right]_{C,1}
= \left[\begin{array}{cccc} 0 & 0 & \cdots & 0 \end{array}\right]^T_{1,\lambda k}
\tag{3.47}
$$

By Theorem (10), presented in Appendix A, a basis for the column space of $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]_{\lambda k,C}$, brought to *rref*, consist of the columns in which the pivots are located. Thus, as indicated in Equation (3.47), the columns containing the boxed ones are pivot columns, which correspond to the leading variables. The set of leading variable form a set of independent constraints of the mechanism, or a basis for the constraint space. The constraints corresponding to the free variables can eventually be eliminated from the mechanism, resulting in a self-aligning mechanism.

It is important to regard that *different basis can be chosen to span the constraint space.* This means that different set of redundant constraints can be eliminated in order to turn the overconstrained mechanism into a self-aligning one. Although the *rref* of a matrix is unique, as stated in Theorem (3) in Appendix A, a different labelling of the mechanism brings to a new matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]_{\lambda k,C}$ and a set of free variables different from the previous one.

This new method, original contribution of this Thesis, is an effective solution for automatic elimination of redundant constraints, which is the case of CAD environment for dynamic simulation. When different candidate sets of redundant constraints to be eliminated need to be rated, which is mostly the case of designing of a novel mechanism or analysis an existing one, the new method proposed in Chapter 4.1 can be employed.

3.3.1.1 Single-loop spatial mechanism: actions analysis

The mechanism of Figure 24 is considered. For the single loop mechanism, the unit action matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{D}}\right]$ and the network unit action matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$ are presented in Appendix C.1. The cutset law can be written as Equation (C.14). Matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$ can thus conveniently be written in *rref* form, as in Equation (C.15). Thus the columns containing the boxed ones are pivot columns, which correspond to the leading variables.

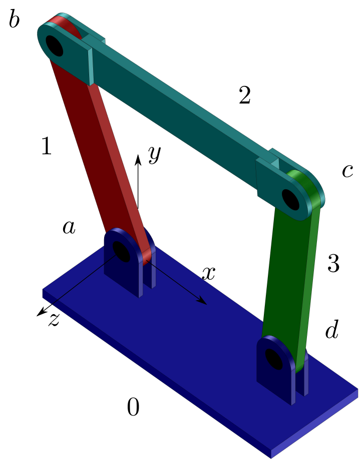


Figure 24 – Single-loop spatial mechanism, $F_N = 1$ and $C_N = 3$.

Therefore a self-aligning mechanism, kinematically equivalent to the one presented in Figure 24, is described by the set:

$$\text{Self-aligning set:} = \{a_R, a_S, a_U, a_V, a_W, b_R, b_S, b_U, b_V, b_W, c_R, c_S, c_U, c_V, c_W, d_U, d_V\} \tag{3.48}$$

In this mechanism, three additional freedoms have been added joint d, more exactly two angular mobilities around x and y-axis and a linear mobility along z-axis.

It can be regarded that the redundant constraints already computed in the previous sections for the four-bar mechanism, *i.e.* torques around x and y axis and a force along z-axis, have been eliminated by removing the corresponding constraints in joint d.

The self-aligning mechanism obtained in set (3.48) is not unique. In Chapter 4 a new method is proposed for enumerating all possible self-aligning mechanisms kinematically equivalent to a given one. The self-aligning corresponding to set (3.48) is presented in Figure 25, where in coupling d three constraints, R,S and W, have been eliminated.

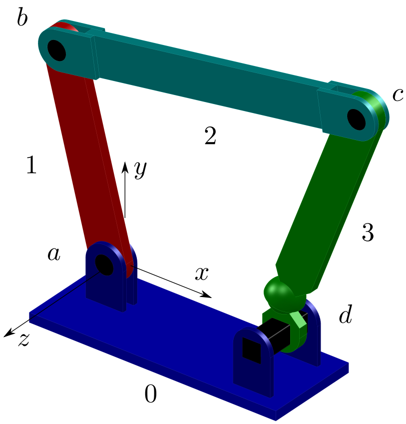


Figure 25 – Self-aligning mechanism derived from the mechanism in Figure 24

3.4 NEW METHOD FOR AUTOMATIC ELIMINATION OF REDUNDANT CONSTRAINTS

In this section a new method, original contribution of this Thesis, for automatic elimination of redundant constraints is presented. In engineering simulations of multibody systems often moving bodies are modelled as rigid. Neglecting elasticity permits faster and simpler simulations, simply taking into account masses and main dimensions of bodies, and ignoring the joint reactions, when redundant constraints exist in the given mechanism. For a given mechanism, the number of equations in its rigid body model is much lower than its elastic body model.

In general rigid body assumption leads to more effective simulations. On the other hand, one of the main rigid body limitation is the indeterminacy of joint reactions, when redundant constraints exist in the given mechanism. When a mechanism is overconstrained it is statically indeterminate. This means that the number of variables are greater than the number of static equations, and constitutive equations must be considered in order to solve reactions. A simple example is introduced. In Figure 26 a planar structure, *i.e.* a planar mechanism with zero mobility $F_N = 0$ is presented.

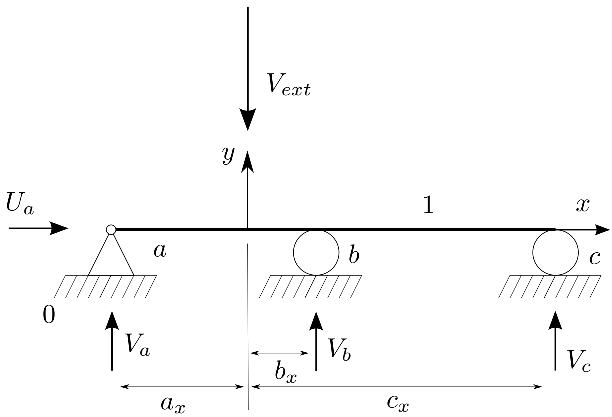


Figure 26 – Hyperstatic structure with $F_N = 0$ and $C_N = 1$

The mechanism comprises two links 0 and 1 and three coupling a, b and c. Coupling a is a revolute one, while couplings b and c are point contact pairs, which constraint only translation along y-axis, allowing rotation along z-axis and translation along x-axis. The equilibrium equations can be written as:

$$\begin{aligned}
\sum U =&0 \quad\rightarrow\quad U_a = 0\\
\sum V =&0 \quad\rightarrow\quad V_a - V_{ext} + V_b + V_c = 0\\
\sum T =&0 \quad\rightarrow\quad V_a \cdot a_x + V_b \cdot b_x + V_c \cdot c_x = 0
\end{aligned} \tag{3.49}$$

Since there are four unknowns U_a, V_a, V_b and V_c and three equilibrium equations, the system of equations does not have a unique solution; The structure is thus statically indeterminate, *i.e.* it has a redundant constraint. Considerations in the material properties and compatibility in deformations are taken into account to solve statically indeterminate structures.

If the focus of a simulation is on kinematics and friction is neglected, it is not necessary to calculate joint reaction forces, and the simulated motion is unique even if joint reactions are indeterminate. On the other hand if friction is considered in an overconstrained mechanism, the simulated motion may not be unique (35).

Thus the problem of determining if joint reactions are solvable or unsolvable is of great importance in software simulation of mechanisms. It should be emphasised that in great majority of general purpose multibody simulation packages (72), (33) detailed information about the state of overconstraint of the system is unavailable.

Two different approaches are employed for handling redundant constraints in multibody systems simulation. The first one consists in modifying the set of equations which describes the mechanism in order to eliminate dependent equations (73). Elimination of redundant constraints is an example of this approach. The second one consists in applying algorithms capable of dealing with dependent equations, leaving the system of equations which describes the mechanism unchanged. Two examples of this approach are the pseudo-inverse-based calculations (74) and the augmented Lagrangian formulation (75). In the pseudo-inverse approach redundant constraint equations are preserved in the mathematical model of a mechanism. The motion equations form a system with multiple solutions. It is possible to solve this type of systems using the Moore-Penrose pseudo-inverse matrix (74). In the augmented Lagrangian formulation (75) the constraint equations are incorporated as a dynamical system, penalized by a large factor, into the equations of motion. In this way the original problem given by equations of motion and constraint equations is replaced by an optimization problem.

Regarding the first approach, a general form of eliminating redundant constraints can be performed mathematically: only the subset of independent equations is considered. The redundant constraints elimination can be done manually selecting the redundant constraints, or automatically. One common method of automatic elimination of redundant (implemented in widely used multibody software) is based on the constraint Jacobian matrix analysis (73). In this method, all constraint equations are formulated and then they are divided into subsets of independent and dependent equations. The equations classified as dependent are excluded from the mathematical model. It is important to regard that, regardless the method employed for redundant constraint elimination, the reaction forces associated with eliminated constraints are set to zero. This implies that the load of the eliminated constraints are transferred to the remaining ones. Thus redundant constraints elimination affects not only reactions of eliminated constraints, but modify as well the reactions of remaining constraints. The selection of redundant constraints is not unique, thus different set of redundant constraints can be selected for elimination.

In this section a new method for handling and for automatic elimination of redundant constraints is proposed, based on the results introduced in the previous sections and on the representation of mechanisms in terms of freedoms and constraints.

3.4.1 A new method for overconstraint analysis

A new method for automatic elimination of redundant constraints is herein presented, based on the result obtained in Section 3.3.1.

Algorithm 1 Automatic elimination of redundant constraints

1: Create the graph G_c of the given mechanism;
2: Create the unit action matrix $\hat{\boldsymbol{A}}_{\boldsymbol{D}}$;
3: Create Action graph G_A;
4: Create the cut matrix Q_A;
5: Create the network unit action matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$ for the given mechanism;
6: Calculate the degree of redundant constraints C_N of the mechanism
7: **if** $C_N \neq 0$ **then**
8: Bring matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$ into *rref* form
9: Eliminate the set of redundant constraints corresponding to the free variables, as indicated in Section 3.3.1.

Thus the set of constraints corresponding to pivot columns form a self-aligning mechanism, kinematically equivalent to the given mechanism. The method presented can be added to CAD software for increasing software capability of dealing with overconstraint.

In order to analyse the complexity of Algorithm (1), the worst case in terms of matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$ size is analysed. Given a spatial mechanism with n links and g joints, matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$ has size $\lambda k \times C$, with $C = \sum c_i$ where c_i is degree of constraint of coupling i and k is the number of fundamental cut sets. The number of fundamental cut sets for a graph is $k = n-1$ (76), and for each joint the maximum degree of constraint, *i.e.* the maximum number of actions transmitted, is $c_i = 5$. Thus, for a spatial ($\lambda = 6$) mechanism with n links and g joints, matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$ has size $6 \cdot (n-1) \times 5 \cdot g$ in the worst case.

The *rref* algorithm, for a matrix with size $a \times b$ has complexity $O(a^2 b)$ (77). Therefore, for a given mechanism with n links and g joints, the worst case for the Algorithm (1) is $O((6 \cdot (n-1))^2 (5 \cdot g))$.

The correctness of Algorithm (1) can be shown in the following form: Steps $1-6$ implement the Davies' method, well established in literature: for a given mechanism, matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$ can always be obtained (48, 17). Any finite matrix can always be reduced in many ways by a finite sequence of *ERO* to *rref* form (78), thus the existence of the *rref* form is granted for Step 8. In Step 9 the elimination of redundant constraints corresponding to the free variables, can be performed.

3.4.1.1 Elimination of redundant constraints for a planar mechanism

In this section, an application of the Algorithm 1, introduced in the previous section, is presented. The investigated mechanism is depicted in Figure 27.

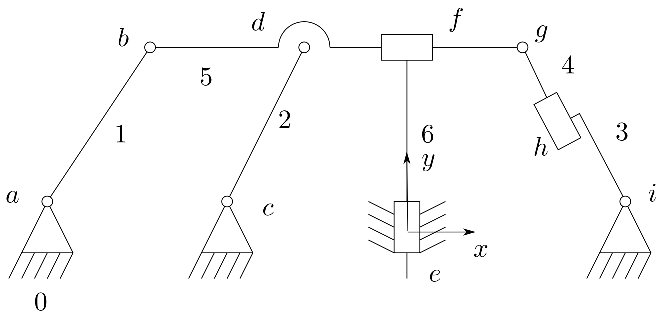


Figure 27 – Planar mechanism with $F_N = 1$ and $C_N = 1$

This mechanism has $F_N = 1$ and $C_N = 1$. It has been analysed by Wojtyra, Frä et al.(33) by means of constraint Jacobian formulation. In this section the analysis with the new Algorithm (1) is presented.

The graph G_M of the mechanism is shown in Figure 28.

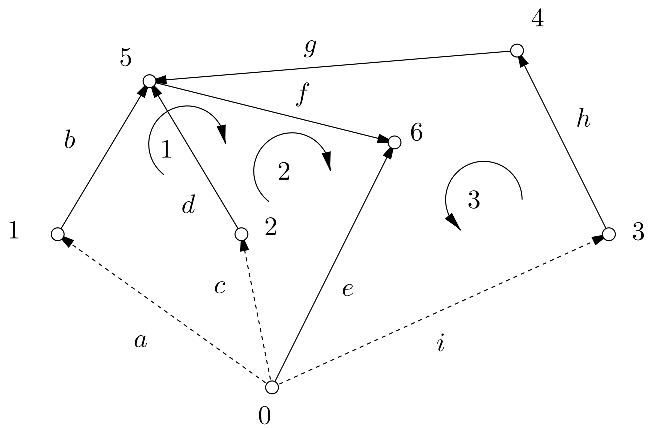


Figure 28 – Graph of mechanism in Figure 27

Thus the circuit matrix $\boldsymbol{B_M}$ and the cutset matrix $\boldsymbol{Q_A}$ can be written as:

$$[\boldsymbol{B_M}] = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \end{array} \begin{array}{c} \begin{array}{ccccccccc} a & b & c & d & e & f & g & h & i \end{array} \\ \left[\begin{array}{ccccccccc} 1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 1 \end{array}\right]_{3,9} \end{array} \tag{3.50}$$

$$[\boldsymbol{Q_A}] = \begin{array}{c} \\ k_b \\ k_d \\ k_e \\ k_f \\ k_g \\ k_h \end{array} \begin{array}{c} \begin{array}{ccccccccc} a & b & c & d & e & f & g & h & i \end{array} \\ \left[\begin{array}{ccccccccc} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right]_{6,9} \end{array} \tag{3.51}$$

The coupling coordinates are:

	a	**b**	**c**	**d**	**e**	**f**	**g**	**h**	**i**
x	-6	-4	-4	-2	0	0	2	3	4
y	0	2	0	2	0	2	2	1	0

and matrix $\boldsymbol{\hat{M}_D}$ can be written as (matrix $\boldsymbol{\hat{M}_D}$ is presented only for completeness, it is not necessary for Algorithm 1 application):

$$\left[\boldsymbol{\hat{M}_D}\right] = \begin{array}{c} \\ t \\ u \\ v \end{array} \begin{array}{c} \begin{array}{ccccccccc} a & b & c & d & e & f & g & h & i \end{array} \\ \left[\begin{array}{ccccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 & 0 & 1 & 2 & -\frac{\sqrt{2}}{2} & 0 \\ 6 & 4 & 4 & 2 & 1 & 0 & -2 & \frac{\sqrt{2}}{2} & -4 \end{array}\right]_{3,9} \end{array} \tag{3.52}$$

Matrices $\boldsymbol{\hat{A}_D}$ and $\boldsymbol{\hat{M}_N}$ are presented in Section C.2. The degree of redundancy of the mechanism can be calculated as $C_N = C - rank(\boldsymbol{\hat{M}_N}) = 18 - 17 = 1$. Thus the mechanism has one redundant constraint. Matrix $\boldsymbol{\hat{M}_N}$ is brought in *rref* form, as in Equation (C.18). Constraint f_V, *i.e.* the constraint of coupling f along y-axis is a redundant constraint. The elimination of this constraint by introducing a freedom along y-axis in coupling f turn the mechanism of Figure 27 into a self-aligning mechanism. The result obtained applying Algorithm 1 is in accordance with the analysis performed by Wojtyra, Frä et al.(33).

4 ANALYSIS OF OVERCONSTRAINED MECHANISMS BY MEANS OF MATROID THEORY

In this chapter a further analysis based on the network unit matrices $\hat{\boldsymbol{M}}_{\boldsymbol{N}}$ and $\hat{\boldsymbol{A}}_{\boldsymbol{N}}$ is presented. The linear dependence and independence of freedoms and constraints in a given mechanism can be successfully explored by the application of matroid theory. A brief introduction to the concept of matroid is first presented. Then a new and original approach, based on matroid theory, is applied for solving two different problems of mechanism: enumeration and selection of valid actuation schemes, enumeration and selection of self-aligning mechanism kinematically equivalent to a given mechanism. A set of new algorithms, original contributions of this Thesis, are therefore presented.

4.1 MATROIDS

The concept of matroid is a combinatorial abstraction of matrices with respect to linear independence. It was first introduced by Hassler Whitney(79) as an abstraction of linear independence. Many alternative formulations of matroids can be found in literature (80), where multitude of non-obviously equivalent definitions goes by the name of cryptomorphism. In this Thesis, the matroid theory developed from linear algebra is employed. Formal definition and a review of the main properties are presented in Appendix A, herein an example is presented illustrating the basic structure of a linear matroid.

Given matrix A:

$$A = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}_{3,5} \end{array} \tag{4.1}$$

whose columns are indexed by $E = \mathrm{Col}(A) = \{1, \ldots, 5\}$, the linear dependence and independence among column vectors $\{\boldsymbol{a_e} | e \in E\}$ is considered. A subset $I \subseteq E$ is said to be independent if the corresponding subfamily $\{\boldsymbol{a_e} | e \in I\}$ of column vectors is linearly independent. The family of independent subsets, denoted by $\mathcal{I} \subseteq 2^E$, *i.e.* all possible subsets of E, is defined as:

$$\mathcal{I} = \{I \subseteq E | \{\boldsymbol{a_e} | e \in I\} \text{ is linearly independent}\} \tag{4.2}$$

For the matrix A of Equation (4.1):

$$\begin{aligned}\mathcal{I} =& \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\},\\ & \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}, \{1,2,3\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\},\\ & \{1,4,5\}, \{2,3,4\}, \{2,4,5\}, \{3,4,5\}\}.\end{aligned} \tag{4.3}$$

Each one of the sets of Equation 4.3 represents a linear independent set. Note that the set of columns $\{1, 2, 4\} \notin \mathcal{I}$, because column 4 can be obtained a linear combination of columns 1 and 2. The following three properties can be applied to $\mathcal{I}$:

1. $\emptyset \in \mathcal{I}$
2. If $I \in \mathcal{I}$ and $J \in \mathcal{I}$ than $I \subseteq J \in \mathcal{I} \Longrightarrow I \in \mathcal{I}$
3. $I, J \in \mathcal{I}, |I| < |J| \Longrightarrow I \cup \{v\} \in \mathcal{I} \quad \text{for some} \quad \nu \in J \setminus I$

Property (2) states that all subsets of an independent set are independent sets. For example, set $\{1, 2, 3\} \in \mathcal{I}$ is an independent set thus subsets $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$ are all independent sets, as stated in Equation (4.3). Because of property (2), it is redundant to enumerate all the members of $\mathcal{I}$. Instead the family $\mathcal{B}$ of the maximal members, *i.e.* the set with maximal cardinality, of $\mathcal{I}$ is sufficient:

$$\begin{aligned}\mathcal{B} =& \{\{1,2,3\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \{2,3,4\}, \{2,4,5\},\\ & \{3,4,5\}\}\end{aligned} \tag{4.4}$$

which represent the family of bases of the column space of matrix $\boldsymbol{A}$.

Another way of representing the independence structure of column vectors is given by the rank function $r : 2^E \to \mathbb{Z}$:

$$r(X) = \text{rank } A([\text{Row}(\boldsymbol{A}), X]), \qquad \text{with} \quad X \subseteq V \tag{4.5}$$

where $([\text{Row}(\boldsymbol{A}), X])$ is the set of X columns of $\boldsymbol{A}$ and $r(X)$ can be also regarded as the dimension of the vector space spanned by the X columns of $\boldsymbol{A}$. A set of properties equivalents to (4.3), stated for the family of independent subsets $\mathcal{I}$, can also be stated for the rank function $r(X)$.

Thus a matrix gives rise to pairs $(E, \mathcal{I})$, $(E, \mathcal{B})$ and (E, r), each representing the linear independence of linear structure of column vectors. $(E, \mathcal{I})$, $(E, \mathcal{B})$ and (E, r) are equivalent and define the same combinatorial structure underlying linear independence. This structure is named matroid (81) and E is called the ground set of the matroid.

4.2 ACTUATION SCHEMES ENUMERATION AND SELECTION

Given a mechanism, there are many different ways of selecting the actuated joints in order to control the whole mechanism, but in general the actuated joints cannot be chosen arbitrarily. For a mechanism with mobility F_N, F_N actuated joints should be selected in order to control the whole mechanism. Assume that for some mechanisms it is possible to control the end-effector with a number of actuators smaller than F_N, leaving in this case a part of the mechanism uncontrolled. Along this Thesis, actuation of the whole mechanism is considered.

The selection of actuated joints should ensure that, in a general configuration, the left mobility of the mechanism, when the F_N actuated joints are blocked, is zero. This statement is true when non redundant actuation is considered and all links are controlled. In this Thesis, only non-redundant actuation is considered.

The problem of correctly selecting a set of joints in a given mechanism has been addressed by various authors. In (82) linear transformation for structural synthesis of parallel mechanisms are applied, and a set of conditions is stated which express the possibility of direct actuation of each independent motion of the mobile platform by just one specific actuator.

In (12) the number of possible actuation schemes is calculated with a combinatorial formula, based on the mobility. Each actuation scheme is then verified through statics analysis. Zhao et al.(12) point out that their method can only provide the number of actuations needed to control all the links in the mechanism, but not guarantee that any actuation scheme is available to control the end effector. In some classes of mechanism the number of actuators needed to control the end-effector is different from the number needed to control the whole mechanism. Zhao et al.(18) apply this method for a *3-PPRR* mechanism, shown in Figure 29. The same mechanism is analysed in this thesis with an original method, as presented in the next sections.

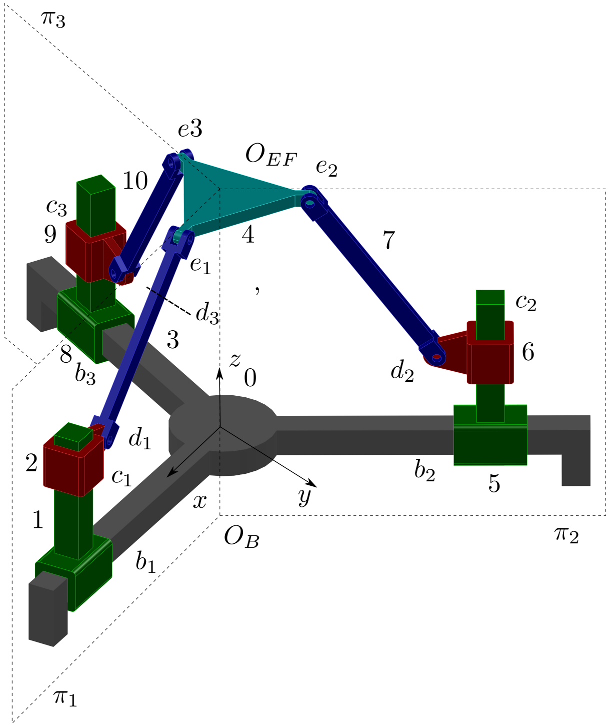


Figure 29 – Spatial 3-PPRR parallel mechanism analysed

Qin and Dai(19) investigate a kind of *2US+UPS* asymmetrical parallel mechanism, presented in Figure 30. Mobility analysis via screw theory proved that the asymmetrical parallel mechanism has 4-DOF coupling with three rotations and one translation. Accordingly, two kinds of actuated strategies of the mechanism have been designed: the prismatic joint in the first limb and the first revolute joint parallel in each limb were chosen as one kind of actuated strategy.

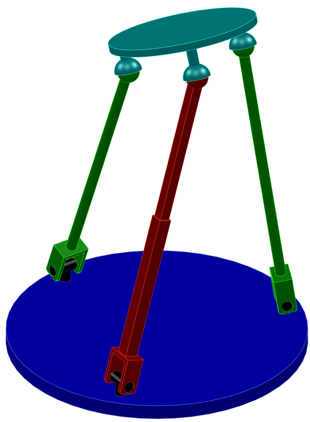


Figure 30 – *2US+UPS* asymmetrical parallel mechanism

Gan et al.(20) present the analysis of a new metamorphic parallel mechanism, showed in Figure 31, with four topologies by altering the limb phases with mobility *1R2T*(one rotation with two translations), *2R2T*, *3R2T* and mobility 6. Actuation schemes are discussed by covering all the topologies of the metamorphic parallel mechanisms based on constraint screws.

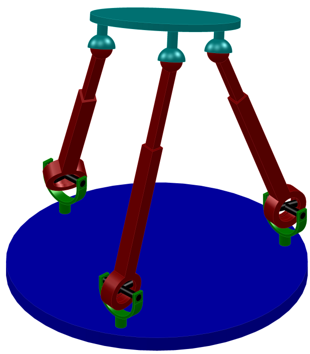


Figure 31 – Metamorphic parallel mechanism

Ebrahimi, Carretero and Boudreau(21) present the study of two different manipulators, one non-redundant (*3-RRR*) and the other kinematically redundant (*3-PRRR*). Applying two distinct methods, the actuation schemes for both manipulator are obtained. The results show that using kinematic redundancy significantly improves the characteristics of the manipulator as it can avoid singularities.

Matone and Roth(22) study the modelling of actuation schemes and their effects on the singularities of parallel manipulators. Necessary and sufficient conditions on how to transform a singular configuration into a non-singular one by changing actuators locations are presented. The results are applied to a five-bar parallel manipulator.

Kong, Gosselin and Richard(15) state a validity condition of actuated joints in parallel mechanisms. First the set of all the action screws which are not reciprocal to the motion screws of joint j and reciprocal to all the motion screws of the other joints within leg i are referred as $W^i_{\nsupseteq j}$ with $(j = 1, 2, \ldots, c_i)$.

The set $W^i_{\nsupseteq j}$ can be regarded as the set of action screws that can be exerted on the moving platform through the actuation of j in leg i. Then $\zeta^i_{\nsupseteq j}$ is defined as any one arbitrary action screw which belongs to $\mathcal{W}^i_{\nsupseteq j}$. $\boldsymbol{\zeta}^{\boldsymbol{i}}_{\nsupseteq \boldsymbol{j}}$ is called the *actuation wrench or actuation screw* of joint j in leg i. For example, the leg i, with $PRRR$ couplings, presented in Figure 32 is considered, where the prismatic coupling is actuated. The actuation screw of the prismatic pair for leg i is thus the action screw $\boldsymbol{\zeta}^{\boldsymbol{i}}_{\nsupseteq \boldsymbol{j}}$ whose axis is parallel to the axes of the three revolute joints within the same leg.

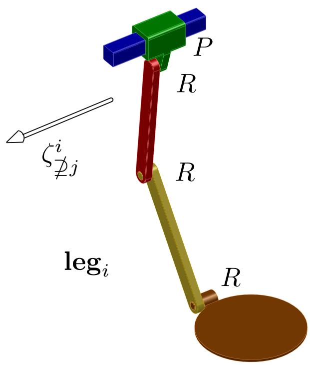


Figure 32 – Actuation force of (PRRR) leg

Thus for a given parallel manipulator with F_N degrees of freedom, a given set of actuated joints is valid if and only if $F'_N = 0$, where $F'_N = 0$ denotes the mobility of the parallel manipulator with all its actuated joint blocked. This means that the action screw system of a parallel manipulator with its actuated couplings blocked, can be regarded as a linear combination of the action screw system W of the mechanism and all actuation screws $\boldsymbol{\zeta^{i}_{\not\supset j}}$ of the actuated joints. Kong, Gosselin and Richard(15) state this condition in the form:

Condition 1. *For any parallel manipulator with dof F_N, in which all the motion screws within the same leg are linearly independent in a general configuration, a set of F_N actuated joints is valid if and only if, in a general configuration, all the actuation screws, $\boldsymbol{\zeta^{i}_{\not\supset j}}$, of the N actuated joints together with a set of basis action screws of the wrench system $\mathcal{W}$ of the parallel kinematic chain constitute a 6-system. $\mathcal{W}^{i}_{\not\supset j}$ is the set of all the action screws which are not reciprocal to the motion screw of joint j and reciprocal to all the motion screws of the other joints within leg i.*

For a better understanding the mechanism *Tripteron*, already presented in Figure 21, is considered. The wrench system of the platform (link 4) can be intuitively obtained regarding that limb $a-b-c-d$ imposes to the platform two constraints, more exactly a torque around y-axis and a torque around z-axis. Thus the wrench system of the limb can be written as:

$$\{W_{a-b-c-d}\} = \left\{ \begin{array}{c} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T \end{array} \right\} \tag{4.6}$$

In the same way, the wrench systems of limbs $e-f-g-h$ and $i-l-m-n$ can be respectively written as:

$$\{W_{e-f-g-h}\} = \left\{ \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T \end{array} \right\} \tag{4.7}$$

$$\{W_{i-l-m-n}\} = \left\{ \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T \end{array} \right\} \tag{4.8}$$

The wrench system of the platform can eventually be obtained as:

$$\begin{aligned} \{W_{\text{platform}}\} &= W_{a-b-c-d} \cup W_{e-f-g-h} \cup W_{i-l-m-n} \\ &= \left\{ \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T \end{array} \right\} \end{aligned} \tag{4.9}$$

The actuation screws of a valid actuation scheme must therefore constitute a 6-system with the wrench system of Equation (4.9). If the three prismatic pairs a, e and i are actuated, the actuation screws are respectively:

$$\begin{aligned}
\boldsymbol{\zeta}^{\boldsymbol{a}}_{\not\supseteq(\boldsymbol{a-b-c-d})} &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T \\
\boldsymbol{\zeta}^{\boldsymbol{e}}_{\not\supseteq(\boldsymbol{e-f-g-h})} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T \\
\boldsymbol{\zeta}^{\boldsymbol{i}}_{\not\supseteq(\boldsymbol{i-l-m-n})} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T
\end{aligned} \tag{4.10}$$

Assume that the same result of Equation 4.10 applying reciprocal screw theory (12, 13, 83).

Thus the actuation scheme $\{a, e, i\}$ is a valid actuation scheme for the *tripteron* mechanism, because:

$$\dim(W_{platform} \cup \boldsymbol{\zeta}^{\boldsymbol{a}}_{\not\supseteq(\boldsymbol{a-b-c-d})} \cup \boldsymbol{\zeta}^{\boldsymbol{e}}_{\not\supseteq(\boldsymbol{e-f-g-h})} \cup \boldsymbol{zeta}^{\boldsymbol{i}}_{\not\supseteq(\boldsymbol{i-l-m-n})}) = 6 \tag{4.11}$$

Thus, this thesis focuses on the actuation which controls all links of a given mechanism, with no redundant actuation. Starting from the circuit law modelling stated by Davies, presented in Section 2.2.1, a novel method is herein proposed for enumerating all valid actuation schemes for a given mechanism. Moreover, a new method for selection of an optimal actuation scheme is proposed, based on classification of the couplings.

Given a mechanism, the circuit law, can be written as:

$$\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]_{\lambda\nu,F} [\boldsymbol{\psi}]_F = [\mathbf{0}]_{dl} \tag{4.12}$$

where $\boldsymbol{\psi}$ is the vector of F generalised motion magnitudes and $\hat{\boldsymbol{M}}_{\boldsymbol{N}}$ is the network unit motion matrix. Equation 4.12 represents a linear homogeneous system, which allows solving the unknown motion magnitudes in terms of a set of primary variables. As introduced in Section 2.2.1, the set of primary variables describes the state of freedom of the mechanism, and the number of primary variables F_N is the degree of freedom, or mobility, of the mechanism (17).

Also F_N indicates the number of actuators needed in order to control the whole mechanism, when no redundant actuation is considered. Furthermore, actuators cannot be assigned to any set of couplings, but must be assigned to a specific (but not unique in general) set of couplings. This set of coupling to be actuated must be a set of primary variables, in order to control the whole mechanism without leaving any part of the mechanism under-actuated or over-actuated.

An example may be useful for an intuitive understanding of this statement, before introducing a formal proof. The two-loop planar mechanism presented in Figure 33 has 7 links (the link 0 is regarded as basis), 8 couplings and it presents mobility $F_N = 2$. Two actuators are needed in order to control the mechanism. Intuitively, each symmetric part of the mechanism presents a mobility. Moreover, for each loop, the positions of the three links relative to the basis 0 are determined by the single actuator present in the loop. In other words, each one of the two set of links and joints $\{a, 1, b, 2, c, 3, d\}$ and $\{e, 4, f, 5, g, 6, h\}$, is completely and independently defined in terms of one parameter. For instance, any of the couplings belonging to the set $\{a, b, c, d\}$ can be selected as active coupling in order to control links $\{1, 2, 3\}$, while the position of links $\{4, 5, 6\}$ depend exclusively on one active coupling selected in the set $\{h, g, f, e\}$. This kind of mechanism is called *fractionated*. A kinematic chain is classified as fractionated if the elimination of a single element of the kinematic chain (body or joint) divides the kinematic chain into two disconnected kinematic chains. Otherwise it is non-fractionated. Fractionation can be classified into three types: body fractionation, joint fractionation and fractionation into hybrid kinematic chains (84). The mechanism show in Figure 33 is body fractionated.

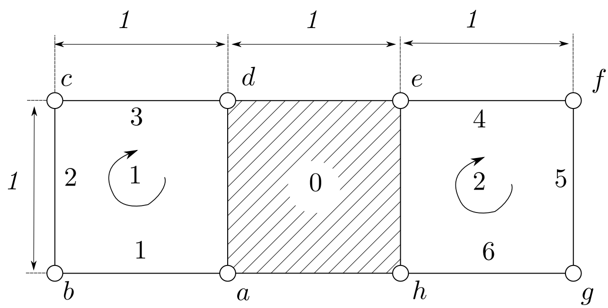


Figure 33 – Two-loop planar mechanism with mobility $F_N = 2$ and $C_N = 0$.

Thus different pairs of couplings may be selected to be actuated, but in order to control the whole mechanism, each loop must contain only one actuator. If for instance, couplings a and b are selected as active couplings, loop 2 is left completely uncontrolled, *i.e.* one freedom is left in the mechanism actuated. On the other hand, in loop 1 the two active couplings concurrently actuate the single free motion of the loop.

The selection of feasible actuation schemes can be regarded as a problem of evaluating linear dependence and independence. The network unit motion matrix $\hat{\boldsymbol{M}}_{\boldsymbol{N}}$ captures the essential relations between couplings. Matrix $\hat{\boldsymbol{M}}_{\boldsymbol{N}}$ presents the following structure:

$$
\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]_{\lambda\nu,F} =
\begin{array}{c}
\begin{array}{cccc}
\overbrace{\qquad\qquad}^{\text{Coupling a}} & \overbrace{\qquad\quad}^{\text{Coupling b}} & \overbrace{\quad\;}^{\dots} & \overbrace{\quad\;}^{\dots}
\end{array} \\
\left[\begin{array}{ccc:cc:c:cc}
* & * & \dots & * & \dots & \dots & \dots & * \\
* & * & \dots & * & \dots & \dots & \dots & * \\
 & & & & & & & \\
* & * & \dots & * & \dots & \dots & \dots & * \\
 & & & & & & & \\
* & * & \dots & * & \dots & \dots & \dots & * \\
* & * & \dots & * & \dots & \dots & \dots & * \\
 & & & & & & & \\
* & * & \dots & * & \dots & \dots & \dots & * \\
 & & & & & & & \\
* & * & \dots & * & \dots & \dots & \dots & * \\
* & * & \dots & * & \dots & \dots & \dots & * \\
 & & & & & & & \\
* & * & \dots & * & \dots & \dots & \dots & *
\end{array}\right]
\end{array}
\begin{array}{l}
\\
\left.\begin{array}{l}
\}\ \text{dim. } 1 \\
\}\ \text{dim. } 2 \\
\qquad\quad\vdots \\
\}\ \text{dim. } \lambda
\end{array}\right\}\ \text{Circuit } 1 \\
\left.\begin{array}{l}
\}\ \text{dim. } 1 \\
\}\ \text{dim. } 2 \\
\qquad\quad\vdots \\
\}\ \text{dim. } \lambda
\end{array}\right\}\ \text{Circuit } 2 \\
\qquad\qquad\qquad\vdots \\
\left.\begin{array}{l}
\}\ \text{dim. } 1 \\
\}\ \text{dim. } 2 \\
\qquad\quad\vdots \\
\}\ \text{dim. } \lambda
\end{array}\right\}\ \text{Circuit } \nu
\end{array}
\tag{4.13}
$$

where each column is uniquely determined by a single freedom (a single *dof*) allowed by a coupling. Dependence and independence of the mechanism's freedoms can thus be analysed by studying the properties of $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]_{\lambda\nu,F}$.

When a given mechanism is completely actuated, *i.e.* all links are controlled and no redundant actuation is present, no degree of freedom, *i.e.* no mobility, is left in the mechanism. Regarding Equation (4.12) this result is achieved when the primary variables are determined by the actuators, and the left unknown magnitudes are thus expressed in terms of the primary variables.

Alternatively, if active couplings are regarded as passive couplings, thus with the corresponding freedoms locked, a new homogeneous linear system expressing the circuit law can be written, where the columns corresponding to the passive variables are removed.

$$\left[\hat{\boldsymbol{M}}'_{\boldsymbol{N}}\right]_{\lambda\nu,F-F_N}\left[\boldsymbol{\psi}'\right]_{F-F_N,1}=[\boldsymbol{0}]_{\lambda\nu,1} \tag{4.14}$$

If the actuation scheme selected is feasible, the new linear system is completely determined and no freedom is left in the mechanism. Thus by Theorem (4) $\left[\hat{\boldsymbol{M}}'_{\boldsymbol{N}}\right]_{\lambda\nu,F-F_N}$ has full rank (rank($\left[\hat{\boldsymbol{M}}'_{\boldsymbol{N}}\right]_{\lambda\nu,F-F_N}$) = $F-F_N$) and the system has only the trivial solution, as expected. The column vectors of $\left[\hat{\boldsymbol{M}}'_{\boldsymbol{N}}\right]_{\lambda\nu,F-F_N}$ are therefore linearly independent. Thus any set of active couplings which leave the corresponding matrix $\left[\hat{\boldsymbol{M}}'_{\boldsymbol{N}}\right]_{\lambda\nu,F-F_N}$ (with the columns corresponding to active couplings removed) with full rank is valid scheme of actuation. Analogously to Condition 1, a new condition, contribution of this Thesis, can now be stated to check the feasibility of a set of actuated joints in a mechanism:

Condition 2. *For a mechanism with F_N degree of freedom in a given configuration, a set of F_N actuated joints is valid if and only if, in the given configuration, the corresponding matrix $\left[\hat{\boldsymbol{M}}'_{\boldsymbol{N}}\right]_{\lambda\nu,F-F_N}$, obtained from $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]_{\lambda\nu,F}$ removing the columns corresponding to the set of active joints, has full rank.*

In general the actuation scheme of a given mechanism is not unique. All combination of $F-F_N$ columns of $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$ can be tested for linear independence in order to find the complementary set of columns which represents a feasible actuation scheme. This method is computationally time-consuming (85).

A new approach, which is an original contribution of this Thesis, is herein proposed. As already introduced, the concept of a matroid is a combinatorial abstraction of matrices with respect to linear independence (81). In the previous chapter, freedoms and constraints in a given mechanism have been described in terms of vectors an matrices, and their relationship in terms of linear dependence and independence. Thus matroids are an elegant and effective structure to capture the relationship between freedoms and constraint in a mechanism. Moreover, they are precisely the hereditary families of sets for which a greedy strategy produces an optimal set. In the next sections, a representation of freedoms and constraints in a mechanism based on matroid theory is introduced. Matroid-based algorithms are then applied in order to analyse some important properties of mechanism.

Given a mechanism and its associated network unit motion matrix $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$, a linear matroid $\mathcal{M}_{M_N}$ is defined over the reals $\mathbb{R}$ from matrix $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$, where the ground set $E(\mathcal{M}_{M_N})$ denotes the index set of the columns of $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]_{\lambda\nu,F}$ $E = \{1, 2, \ldots, F\}$. For a subset X of E, let $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]_{\boldsymbol{X}}$ denote the submatrix of $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$ consisting of those columns indexed by X. A family $\mathcal{I}$ of subset can now be define for the matroid $\mathcal{M}$:

$$\mathcal{I} = \{X \subseteq E : \text{rank}\ ([M_N]_X) = |X|\} \tag{4.15}$$

where a set X is independent if the corresponding columns are linearly independent. A basis B for the matroid $\mathcal{M}$ corresponds to a linearly independent set of columns of cardinality rank $\left(\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]\right) = F - F_N$. Thus the column space of $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$ is spanned by any basis B belonging to the collection of bases of the matroid $\mathcal{B}(\mathcal{M}_{M_N})$. Regarding that a basis B corresponds to a maximal set of independent columns of $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$, the complementary set of B with respect to the ground set E is a set of dependent columns. The corresponding variables of the complementary set are primary variables.

For a generic matroid $\mathcal{M}$ defined on the ground set $E(\mathcal{M})$, a dual matroid $\mathcal{M}^*$ exists on the same ground set $E(\mathcal{M})$. Moreover, $\mathcal{B}^*(\mathcal{M})$ is the set of bases of the dual matroid $\mathcal{M}^*$:

$$\mathcal{B}^*(\mathcal{M}) = \{E(\mathcal{M}) - B \mid \forall B \in \mathcal{B}(\mathcal{M})\} \tag{4.16}$$

Thus considering the matroid $\mathcal{B}(\mathcal{M}_{M_N})$ associated with the mechanism, for any basis B of the matroid, there exists a cobasis B^*, *i.e.* a basis of dual matroid $\mathcal{M}^*_{M_N}$, whose corresponding variables describe a feasible actuation scheme.

A simple example of dual matroid is herein presented for a better understanding. Consider matrix $\boldsymbol{A}$ of equation (4.1), a matroid $\mathcal{M}$ is defined over matrix $\boldsymbol{A}$ as $\mathcal{M}(E, \mathcal{B})$, where the set of bases of the matroid is given by Equation (4.4) and E is the set of columns of matrix $\boldsymbol{A}$. The dual matroid $\mathcal{M}^*$ of the matroid $\mathcal{M}$ is a defined as $\mathcal{M}^*(E, \mathcal{B}^*)$, where the ground set E is the same of the matroid $\mathcal{M}$ and the set of bases of the dual matroid $\mathcal{M}^*$ is:

$$\mathcal{B}^*(\mathcal{M}) = \{\{3,4\},\{2,3\},\{2,4\},\{1,2\},\{1,3\},\{1,5\},\\ \{2,5\},\{4,5\}\} \tag{4.17}$$

The problem of enumerating all feasible actuation schemes of a given mechanism has thus been turned into the problem of enumerating all the bases of dual matroid ${\mathcal{M}^*}_{M_N}$, which is a well-known problem in matroid theory. In next section a new algorithm, based on matroid bases enumeration, is proposed.

4.2.1 New algorithm for enumerating all feasible actuation schemes

A new algorithm for enumerating all feasible actuation schemes of a given mechanism is proposed.

Algorithm 2 Enumerating all feasible actuation schemes for a given mechanism

1: Create the network unit motion matrix $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$;
2: Create the linear matroid $\mathcal{M}_{M_N}$ on the real field $\mathbb{R}$ defined over matrix $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$;
3: Obtain the dual matroid $\mathcal{M}^*_{M_N}$;
4: Enumerate all bases of dual matroid $\mathcal{B}^*(\mathcal{M}_{M_N})$;

Thus sets of variables corresponding to the bases of dual matroid are the feasible actuation schemes of the mechanism. This algorithm has been implemented straightforwardly in the open-source mathematics software system SageMath (86). SageMath is built on an object oriented programming language. It uses this feature to describe categories of mathematical objects, for example algebraic objects like groups, rings and fields, matroids. In SageMath various types of matroids are supported. There are two main entry points to Sage's matroid functionality. The object *matroids* contains a number of constructors for well-known matroids. The function *Matroid()* allows defining matroids from a variety of sources. All classes defining matroids share a common interface, which includes several methods. For the linear matroid class, methods for obtaining the set of bases and cobases of the matroid are available, allowing a straightforward implementation of the algorithm proposed. The reference manual (87) describes the matroid class and the methods available.

In order to analyse the complexity of Algorithm (2), the worst case in terms of matrix $\left[\boldsymbol{\hat{M}_N}\right]$ size is analysed. Given a spatial mechanism with n links and g joints, matrix $\left[\boldsymbol{\hat{M}_N}\right]$ has size $\lambda\nu \times F$, with $F = \sum f_i$ where f_i i the freedom of coupling i and ν is the number of fundamental circuits. The number of fundamental circuits for a graph is $\nu = g - n + 1$ (76), and for each joint the maximum freedom f_i, *i.e.* the maximum number of motions allowed, is $f_i = 5$. Thus, for a spatial ($\lambda = 6$) mechanism with n links and g joints, matrix $\left[\boldsymbol{\hat{M}_N}\right]$ has size $6 \cdot (g - n + 1) \times 5 \cdot g$ in the worst case.

In literature (88), (89) algorithms are proposed for enumerating all bases of a matroid $\mathcal{M}$ in polynomial time $poly(x)$ for base, where x is the cardinality of the ground set E. In the worst case the number of columns for matrix $\left[\boldsymbol{\hat{M}_N}\right]$, *i.e.* the ground set of matroid $\mathcal{M}_{M_N}$, is $|E| = 5 \cdot g$. Thus, each base is enumerated in $poly(5 \cdot g)$. Also the dual matroid $\mathcal{M}^*$ of a matroid $\mathcal{M}$ is obtained in polynomial time. The maximum number of bases for dual matroid $\mathcal{M}^*$ can be therefore estimated. The cardinality of each base is the mobility of the mechanism, *i.e.* $F_N = g - \lambda(n - g - 1) + \sum f_i + C_N$, as stated in Equation (2.24). Thus, in the worst case the number of $\mathcal{M}^*$ bases, *i.e.* the cardinality of $\mathcal{B}^*(\mathcal{M}_{M_N})$, is F_N-combination of the ground set E:

$$|\mathcal{B}^*(\mathcal{M}_{M_N})| = \begin{pmatrix} 5 \cdot g \\ F_N \end{pmatrix} \tag{4.18}$$

The correctness of Algorithm (2) can be shown in the following form: Steps 1 implement the Davies' method, well established in literature: given a mechanism, matrix $\left[\boldsymbol{\hat{M}_N}\right]$ can always be obtained (48, 17). Given a matrix $\left[\boldsymbol{\hat{M}_N}\right]$ a linear matroid $\mathcal{M}$ and its dual $\mathcal{M}^*$ can be defined (90) in Steps $2 - 3$. In Step 4 the bases of dual matroid can be enumerated (89).

In the next section, two example of application of the algorithm are presented. First, the actuation of a simple planar mechanism is analysed and all actuation schemes are enumerated. Then a complex spatial mechanism is considered, and the actuation schemes obtained with the Algorithm (2) are verified with previous results found in literature.

4.2.1.1 Actuation schemes of a planar mechanism

The algorithm for finding all feasible actuation schemes has been applied to the two-loop planar mechanism presented in Figure 33. The unit motion matrix $\left[\hat{\boldsymbol{M}}_{\boldsymbol{D}}\right]$ and the network unit motion matrix $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$ for the mechanism considered are respectively:

$$\left[\hat{\boldsymbol{M}}_{\boldsymbol{D}}\right] = \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \tag{4.19}$$

$$\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right] = \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -0 & -0 & -0 & -0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \tag{4.20}$$

A linear matroid $\mathcal{M}_{M_N}$ over the reals $\mathbb{R}$ is defined over $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$. Thus the ground set of $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$ is formed by the columns of $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$, *i.e.* $E = \{a, b, c, d, e, f, g, h\}$. The complete set of bases of matroid $\mathcal{M}_{M_N}$ is presented in Equation for completeness.

$$
\begin{aligned}
\mathcal{B}(\mathcal{M}_{M_N}) = \{ & \{\ a\ \ b\ \ d\ \ e\ \ f\ \ g\ \}, \{\ b\ \ c\ \ d\ \ e\ \ f\ \ g\ \}, \\
& \{\ a\ \ c\ \ d\ \ e\ \ f\ \ g\ \}, \{\ a\ \ b\ \ c\ \ e\ \ f\ \ g\ \}, \\
& \{\ a\ \ b\ \ d\ \ e\ \ g\ \ h\ \}, \{\ b\ \ c\ \ d\ \ e\ \ g\ \ h\ \}, \\
& \{\ a\ \ c\ \ d\ \ e\ \ g\ \ h\ \}, \{\ a\ \ b\ \ c\ \ e\ \ g\ \ h\ \}, \\
& \{\ a\ \ b\ \ d\ \ f\ \ g\ \ h\ \}, \{\ b\ \ c\ \ d\ \ f\ \ g\ \ h\ \}, \\
& \{\ a\ \ c\ \ d\ \ f\ \ g\ \ h\ \}, \{\ a\ \ b\ \ c\ \ f\ \ g\ \ h\ \}, \\
& \{\ a\ \ b\ \ d\ \ e\ \ f\ \ h\ \}, \{\ b\ \ c\ \ d\ \ e\ \ f\ \ h\ \}, \\
& \{\ a\ \ c\ \ d\ \ e\ \ f\ \ h\ \}, \{\ a\ \ b\ \ c\ \ e\ \ f\ \ h\ \}\}
\end{aligned} \tag{4.21}
$$

The dual matroid $\mathcal{M}^*_{M_N}$ can now be obtained and all sets of cobases of $\mathcal{M}_{M_N}$ are enumerated:

$$
\begin{aligned}
\mathcal{B}^*(\mathcal{M}_{M_N}) = \{ & \{d,h\}, \{a,h\}, \{b,h\}, \{c,h\}, \{c,e\}, \{b,e\}, \{a,e\}, \{d,e\}, \\
& \{c,f\}, \{b,f\}, \{a,f\}, \{d,f\}, \{c,g\}, \{b,g\}, \{a,g\}, \{d,g\}\}
\end{aligned} \tag{4.22}
$$

It can be regarded that each basis of $\mathcal{M}^*_{M_N}$, enumerated in Equation (4.22) is obtained as $E - B(\mathcal{M}_{M_N})$ where E is the ground set and $B(\mathcal{M}_{M_N})$ is a basis of $\mathcal{M}_{M_N}$ in the set of equation (4.21). For example the basis $\{d,h\}$, of dual matroid $\mathcal{M}^*_{M_N}$ is obtained as:

$$
\{d,h\} = \{E - \{\ a\ \ b\ \ c\ \ e\ \ f\ \ g\ \}\} \tag{4.23}
$$

In the same way all bases of dual matroid $\mathcal{M}^*_{M_N}$ can be obtained.

The sets of Equation 4.22 are all feasible actuation schemes for the two-loop planar mechanism of Figure 33. It is important to regard that the two-loop planar mechanism is a *fractioned mechanism*, and each part of the mechanism is a four-bar mechanism, which is controlled by a single actuator. Equation 4.22 contains only one actuated joint from each loop, as expected.

4.2.1.2 Actuation schemes of a spatial single-loop mechanism

The single-loop mechanism of Figure 34 is considered.

The mechanism presented is a four-bar mechanism, with $n = 4$ links and $g = 4$ joints, where a is a planar joint which allows linear

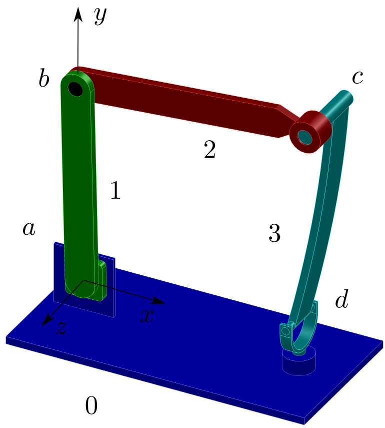


Figure 34 – Single-loop spatial mechanism with mobility $F_N = 3$ and $C_N = 1$.

mobilities along x and y axes and angular mobility around the same axis. Joint b is a revolute coupling, joint c is a cylindrical coupling along z axis and d is an universal coupling along y and z axis.

For the mechanism in Figure 50 matrix $\hat{\boldsymbol{M}}_{\boldsymbol{D}}$, coincident with matrix $\hat{\boldsymbol{M}}_{\boldsymbol{N}}$ as the mechanism is single-loop, cab be written as:

$$\hat{\boldsymbol{M}}_{\boldsymbol{D}} = \hat{\boldsymbol{M}}_{\boldsymbol{N}} = \begin{array}{c} \begin{array}{cccccccc} a_t & a_u & a_v & b_t & c_t & c_w & d_s & d_t \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}_{6,8} \end{array} \tag{4.24}$$

A linear matroid $\mathcal{M}_{M_N}$ over the real $\mathbb{R}$ is defined over $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$. Thus the ground set of $\mathcal{M}_{M_N}$ is formed by the columns of $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$ $E = \{a_t, a_u, a_v, b_t, c_t, c_w, d_s, d_t\}$.

The dual matroid $\mathcal{M}^*_{M_N}$ can now be obtained and all sets of cobases of $\mathcal{M}_{M_N}$ are enumerated:

$$\mathcal{B}^*(\mathcal{M}_{M_N}) = \{\{b_t, c_t, d_t\}, \{a_t, b_t, d_t\}, \{a_t, c_t, d_t\}, \{a_t, b_t, c_t\}, \{a_t, a_u, b_t\}, \{a_t, a_u, c_t\}, \{a_u, b_t, d_t\}, \{a_u, c_t, d_t\}, \{a_u, a_v, b_t\}, \{a_u, a_v, c_t\}, \{a_u, a_v, d_t\}, \{a_t, a_u, a_v\}, \{a_t, a_v, c_t\}, \{a_t, a_v, d_t\}, \{a_v, b_t, d_t\}, \{a_v, b_t, c_t\}\} \tag{4.25}$$

Regard that freedoms c_w and d_s are never actuated, as the closure of the loop inhibit these motions. In terms of linear algebra, it can be verified that columns corresponding to motions c_w and d_s are always independent, thus they cannot be included in the basis of the dual matroid $\mathcal{M}^*_{M_N}$.

4.2.1.3 Actuation schemes of a spatial mechanism *3-PPRR*

The mechanism depicted in Figure 35 is an overconstrained parallel mechanism. It has been proposed by Gogu(82) and again by Zhao (18, 12).

The primary characteristics of this mechanism are that the end-effector 4 is connected to the fixed base 0 through three topologically identical *PPRR* limbs. The two prismatic pairs of each leg, b_i and c_i with $i = 1, 2, 3$, are orthogonal to each other. Intuitively, in each *PPRR* limb the links are always in the same plane π_i. More precisely, for each leg only three independent planar motions are allowed: two translations and one rotation about the normal of the plane. Thus the end-effector is constrained by the three kinematic chains *PPRR* to move along the common intersection of the three planes π_1, π_2 and π_3, *i.e.* the common intersecting line $O_B - O_{EF}$. Thus the connectivity between basis 0 and end-effector 4 is therefore $C_{0,4} = 1$, while the number of actuators needed to control the whole mechanism is $F_N = 4$.

The motion graph G_M of the mechanism is presented in Figure 36.

The circuit matrix $[\boldsymbol{B_M}]_{2,12}$ is:

$$[\boldsymbol{B_M}] = \begin{array}{c} \begin{array}{cccccccccccc} b_1 & c_1 & d_1 & e_1 & b_2 & c_2 & d_2 & e_2 & b_3 & c_3 & d_3 & e_3 \end{array} \\ \left[\begin{array}{cccccccccccc} 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 \end{array}\right]_{2,12} \end{array} \tag{4.26}$$

For the kinematic chain $b_1 - c_1 - d_1 - e_1$ the motion screws associated with the couplings are:

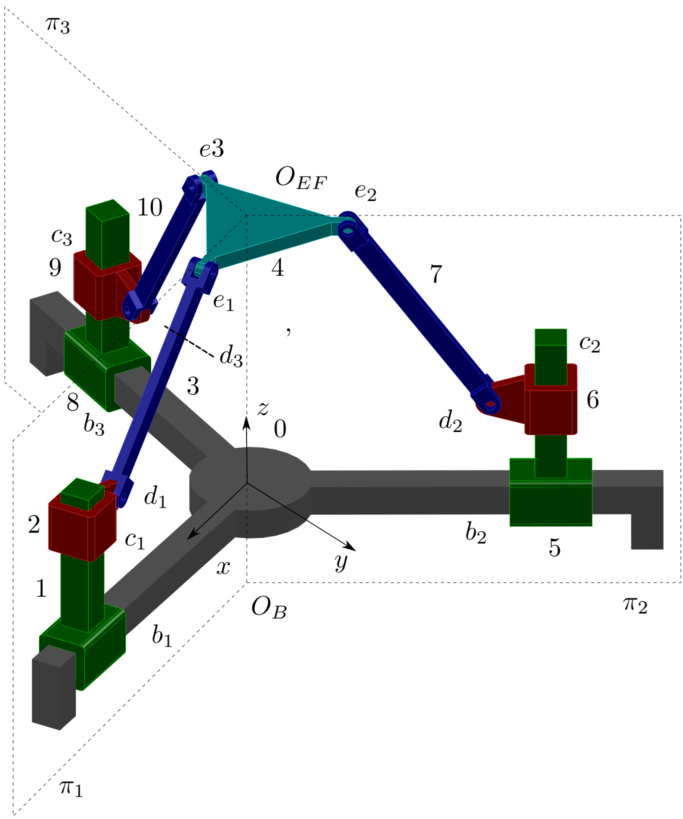


Figure 35 – Spatial 3-PPRR parallel mechanism with mobility $F_N = 4$ and $C_N = 4$.

$$
\begin{aligned}
\hat{\$}^{m}_{b_1} &= \left[\begin{array}{cccccc} 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]^T \\
\hat{\$}^{m}_{c_1} &= \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]^T \\
\hat{\$}^{m}_{d_1} &= \left[\begin{array}{cccccc} 0 & 1 & 0 & -z_{d_1} & 0 & x_{d_1} \end{array}\right]^T \\
\hat{\$}^{m}_{e_1} &= \left[\begin{array}{cccccc} 0 & 1 & 0 & -z_{e_1} & 0 & x_{e_1} \end{array}\right]^T
\end{aligned}
\tag{4.27}
$$

where x_{d_1}, z_{d_1}, x_{e_1} and z_{e_1} are the coordinates of couplings respectively d_1 and e_1 with respect to the global coordinate system $oxyz$ shown in

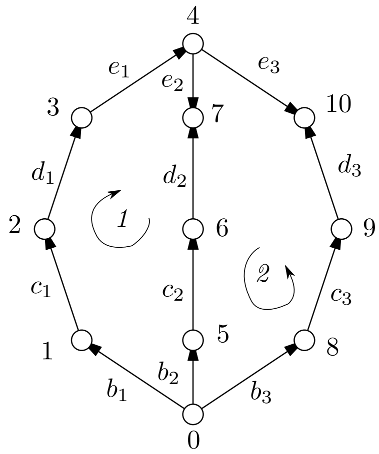


Figure 36 – Graph G_M of 3-PPRR parallel mechanism

Figure 35. Notice that the screw systems associated with the other two kinematic chains $b_2 - c_2 - d_2 - e_2$ and $b_3 - c_3 - d_3 - e_3$ can be obtained from Equation (4.27) by applying a rotation around z-axis of $\frac{2}{3}\pi$ and $\frac{4}{3}\pi$ respectively.

Thus the following linear transformation can be applied:

$$ {}^{\boldsymbol{i}}\boldsymbol{\$} = \left[{}^{\boldsymbol{i}}\boldsymbol{T_j}\right] {}^{\boldsymbol{j}}\boldsymbol{\$} \tag{4.28} $$

which transforms the coordinates of a screw $\boldsymbol{\$}$ from the coordinates system i into the coordinates system j. For the rotation considered around z axis, the transformation matrix can be written as:

$$ \left[{}^{\boldsymbol{i}}\boldsymbol{T_j}\right] = \left[\begin{array}{c|c} \left[{}^{\boldsymbol{i}}\boldsymbol{R_j}\right]_{3,3} & [\mathbf{0}]_{3,3} \\ \hline [\mathbf{0}]_{3,3} & \left[{}^{\boldsymbol{i}}\boldsymbol{R_j}\right]_{3,3} \end{array}\right]_{6,6} \tag{4.29} $$

where $[\mathbf{0}]_{3,3}$ is the rotation matrix around z-axis of an angle θ:

$$ \left[{}^{\boldsymbol{i}}\boldsymbol{R_j}\right] = \begin{bmatrix} \cos(\theta) & \mathrm{sen}(\theta) & 0 \\ -\mathrm{sen}(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3,3} \tag{4.30} $$

Considering a symmetric configuration of the $PPRR$ mechanism, $i.e.$ $z_{d_1} = z_{d_2} = z_{d_3} = 1$, $x_{d_1} = x_{d_2} = x_{d_3} = 2$, $z_{e_1} = z_{e_2} = z_{e_3} = 2$,

$x_{e_1} = x_{e_2} = x_{e_3} = 1$, matrix $\left[\hat{\boldsymbol{M}}_{\boldsymbol{D}}\right]_{6,12}$ and $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]_{12,12}$ can now be written. Both matrices are presented in Appendix C.3.

The rank of $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]_{6,12}$ is rank$(\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]) = 8$, thus the mobility of the *PPRR* mechanism is:

$$F_N = F - rank(\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]) = 12 - 8 = 4 \tag{4.31}$$

and the number of redundant constraints is:

$$C_N = C_N = \lambda\nu - rank(\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]) = 6 \cdot 2 - 8 = 4 \tag{4.32}$$

Applying Algorithm (2) the actuation schemes of the mechanism can be enumerated. A linear matroid $\mathcal{M}_{M_N}$ is defined from matrix $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$, the dual matroid $\mathcal{M}^*{}_{M_N}$ is obtained and the set of bases of dual matroid $\mathcal{B}^*(\mathcal{M}_{M_N})$ can be enumerated.

In order to control the whole mechanism, four actuators are needed. Because of the symmetric structure of mechanism *PPRR* there are some actuation schemes that are the same. Zhao et al. (18, 12) propose a complete list of actuation schemes for this mechanism. First, all possible actuation are generated, considering the following distribution of actuators:

(a) the maximum number of actuators for each kinematic chain is two;

(b) the maximum number of actuators for each kinematic chain is three;

(c) the maximum number of actuators for each kinematic chain is four;

Each group is then split into further subgroups, and all possible selections of actuators are generated applying combinatorial analysis. The actuation schemes which are identical because of the symmetry of the mechanism are eliminated based on cyclic groups. Zhao et al. enumerates a total of possible 98 schemes. Not all of them are able to control the whole mechanism, *i.e.* many of them do not control all links or present redundant actuation. Thus, in (18) **a statics analysis is performed on each scheme in order to verify its feasibility**. A total of 30 feasible actuation schemes are finally proposed by Zhao et al.(18), which control the whole mechanism without redundant actuation.

The Algorithm (2) proposed in this Thesis has **the advantage of enumerating directly all feasible solutions**. In Table 3 all actuation schemes for mechanism *3-PPRR* are presented, as enumerated from the algorithm. A total of 144 feasible schemes is enumerated. Because of the symmetry of mechanism *3-PPRR* some actuation schemes are equivalent and a total of distinct 30 schemes is obtained, exactly the same enumerated by Zhao et al.(18). For this mechanism, the symmetry of the actuation schemes has been checked manually, but for more complex mechanism group theory can be conveniently employed.

In Table 3 the column *"Ac. Set"* indicates a feasible actuation scheme as calculated by Algorithm (2), and the column *"#"* to the side indicates the corresponding actuation scheme enumerated by Zhao et al.(18), with the label reported in his paper. Exactly all the 30 feasible actuation schemes proposed by Zhao et al. are enumerated by the Algorithm (2) proposed in this Thesis.

Table 3 – Enumeration of actuation schemes of mecanism *3-PPRR*

Ac. Set	#	**Ac. Set**	#	**Ac. Set**	#	**Ac. Set**	#
$\{e_1, e_2, c_3, d_3\}$	40	$\{d_1, c_2, c_3, d_3\}$	36	$\{c_1, e_1, e_2, e_3\}$	50	$\{c_1, b_2, b_3, c_3\}$	2
$\{b_1, e_2, c_3, d_3\}$	34	$\{d_1, c_2, e_2, c_3\}$	46	$\{b_1, c_1, e_2, e_3\}$	10	$\{c_1, e_1, b_2, b_3\}$	41
$\{b_1, b_2, c_3, d_3\}$	31	$\{d_1, c_2, e_2, d_3\}$	48	$\{b_1, c_1, b_2, e_3\}$	4	$\{c_1, d_1, b_2, b_3\}$	31
$\{e_1, b_2, c_3, d_3\}$	34	$\{d_1, b_2, c_2, c_3\}$	6	$\{c_1, b_2, c_3, e_3\}$	42	$\{c_1, d_1, e_2, b_3\}$	34
$\{d_1, b_2, c_3, d_3\}$	33	$\{d_1, b_2, c_2, d_3\}$	8	$\{c_1, e_1, b_2, e_3\}$	44	$\{c_1, d_1, d_2, b_3\}$	33
$\{d_1, e_2, c_3, d_3\}$	39	$\{e_1, b_2, c_2, d_3\}$	9	$\{c_1, d_1, b_2, e_3\}$	34	$\{b_1, c_1, d_2, b_3\}$	3
$\{d_1, d_2, c_3, d_3\}$	38	$\{e_1, b_2, c_2, c_3\}$	7	$\{c_1, d_1, e_2, e_3\}$	40	$\{c_1, d_2, b_3, c_3\}$	6
$\{b_1, d_2, c_3, d_3\}$	33	$\{b_1, b_2, c_2, c_3\}$	2	$\{c_1, d_1, d_2, e_3\}$	39	$\{c_1, e_1, d_2, b_3\}$	43
$\{e_1, d_2, c_3, d_3\}$	39	$\{b_1, b_2, c_2, d_3\}$	3	$\{b_1, c_1, d_2, e_3\}$	9	$\{c_1, c_2, d_2, b_3\}$	32
$\{e_1, d_2, c_3, e_3\}$	49	$\{b_1, c_2, c_3, d_3\}$	32	$\{c_1, d_2, c_3, e_3\}$	46	$\{c_1, d_1, c_2, b_3\}$	32
$\{b_1, d_2, c_3, e_3\}$	43	$\{b_1, c_2, e_2, c_3\}$	42	$\{c_1, e_1, d_2, e_3\}$	49	$\{c_1, b_2, c_2, b_3\}$	2
$\{d_1, d_2, c_3, e_3\}$	48	$\{b_1, c_2, e_2, d_3\}$	43	$\{c_1, e_1, d_2, d_3\}$	48	$\{b_1, c_1, c_2, b_3\}$	2
$\{d_1, e_2, c_3, e_3\}$	49	$\{e_1, c_2, e_2, d_3\}$	49	$\{c_1, d_2, c_3, d_3\}$	36	$\{c_1, c_2, e_2, b_3\}$	42
$\{d_1, b_2, c_3, e_3\}$	43	$\{e_1, c_2, e_2, c_3\}$	47	$\{c_1, e_1, d_2, c_3\}$	46	$\{c_1, c_2, b_3, c_3\}$	5
$\{e_1, b_2, c_3, e_3\}$	44	$\{e_1, c_2, c_3, d_3\}$	37	$\{b_1, c_1, d_2, c_3\}$	6	$\{c_1, e_1, c_2, b_3\}$	42
$\{b_1, b_2, c_3, e_3\}$	41	$\{c_1, e_1, c_2, d_3\}$	46	$\{b_1, c_1, d_2, d_3\}$	8	$\{e_1, c_2, b_3, c_3\}$	7
$\{b_1, e_2, c_3, e_3\}$	44	$\{c_1, c_2, c_3, d_3\}$	35	$\{c_1, d_1, d_2, c_3\}$	36	$\{e_1, c_2, e_2, b_3\}$	44
$\{e_1, e_2, c_3, e_3\}$	50	$\{c_1, e_1, c_2, c_3\}$	45	$\{c_1, d_1, d_2, d_3\}$	38	$\{b_1, c_2, e_2, b_3\}$	41
$\{e_1, c_2, c_3, e_3\}$	47	$\{c_1, c_2, e_2, c_3\}$	45	$\{c_1, d_1, e_2, c_3\}$	37	$\{b_1, c_2, b_3, c_3\}$	2
$\{e_1, c_2, e_2, e_3\}$	50	$\{c_1, c_2, e_2, d_3\}$	35	$\{c_1, d_1, e_2, d_3\}$	39	$\{b_1, b_2, c_2, b_3\}$	1
$\{b_1, c_2, e_2, e_3\}$	44	$\{b_1, c_1, c_2, c_3\}$	5	$\{c_1, d_1, b_2, c_3\}$	32	$\{e_1, b_2, c_2, b_3\}$	4
$\{b_1, c_2, c_3, e_3\}$	42	$\{b_1, c_1, c_2, d_3\}$	6	$\{c_1, d_1, b_2, d_3\}$	33	$\{d_1, b_2, c_2, b_3\}$	3
$\{b_1, b_2, c_2, e_3\}$	4	$\{c_1, b_2, c_2, c_3\}$	5	$\{c_1, e_1, b_2, d_3\}$	43	$\{d_1, c_2, e_2, b_3\}$	43
$\{e_1, b_2, c_2, e_3\}$	10	$\{c_1, b_2, c_2, d_3\}$	6	$\{c_1, b_2, c_3, d_3\}$	32	$\{d_1, c_2, b_3, c_3\}$	6
$\{d_1, b_2, c_2, e_3\}$	9	$\{c_1, d_1, c_2, c_3\}$	35	$\{c_1, e_1, b_2, c_3\}$	42	$\{d_1, c_2, d_2, b_3\}$	33
$\{d_1, c_2, e_2, e_3\}$	49	$\{c_1, d_1, c_2, d_3\}$	36	$\{b_1, c_1, b_2, c_3\}$	2	$\{b_1, c_2, d_2, b_3\}$	31
$\{d_1, c_2, c_3, e_3\}$	46	$\{c_1, c_2, d_2, c_3\}$	35	$\{b_1, c_1, b_2, d_3\}$	3	$\{e_1, c_2, d_2, b_3\}$	34
$\{d_1, c_2, d_2, e_3\}$	39	$\{c_1, c_2, d_2, d_3\}$	36	$\{b_1, c_1, e_2, c_3\}$	7	$\{e_1, d_2, b_3, c_3\}$	9
$\{b_1, c_2, d_2, e_3\}$	34	$\{c_1, c_2, d_2, e_3\}$	37	$\{b_1, c_1, e_2, d_3\}$	9	$\{b_1, d_2, b_3, c_3\}$	3
$\{e_1, c_2, d_2, e_3\}$	40	$\{c_1, d_1, c_2, e_3\}$	37	$\{c_1, e_1, e_2, d_3\}$	49	$\{d_1, d_2, b_3, c_3\}$	8
$\{e_1, c_2, d_2, d_3\}$	39	$\{c_1, b_2, c_2, e_3\}$	7	$\{c_1, e_2, c_3, d_3\}$	37	$\{d_1, e_2, b_3, c_3\}$	9
$\{e_1, c_2, d_2, c_3\}$	37	$\{b_1, c_1, c_2, e_3\}$	7	$\{c_1, e_1, e_2, c_3\}$	47	$\{d_1, b_2, b_3, c_3\}$	3
$\{b_1, c_2, d_2, c_3\}$	32	$\{c_1, c_2, e_2, e_3\}$	47	$\{c_1, e_2, b_3, c_3\}$	7	$\{e_1, b_2, b_3, c_3\}$	4
$\{b_1, c_2, d_2, d_3\}$	33	$\{c_1, c_2, c_3, e_3\}$	45	$\{c_1, e_1, e_2, b_3\}$	44	$\{b_1, b_2, b_3, c_3\}$	1
$\{d_1, c_2, d_2, c_3\}$	36	$\{c_1, e_1, c_2, e_3\}$	47	$\{b_1, c_1, e_2, b_3\}$	4	$\{b_1, e_2, b_3, c_3\}$	4
$\{d_1, c_2, d_2, d_3\}$	38	$\{c_1, e_2, c_3, e_3\}$	47	$\{b_1, c_1, b_2, b_3\}$	1	$\{e_1, e_2, b_3, c_3\}$	10

In Appendix C.3 the identical actuation schemes enumerated in Table 3 are grouped into the 30 distinct configurations, taking into account the symmetry of the mechanism.

The following points of the new algorithm herein introduced can be considered:

1. Only feasible actuation schemes are enumerated, thus a statics analysis (18) performed on each actuation is no more necessary, as it is implicit in the matroid basis formulation;

2. The algorithm enumerate each scheme in polynomial time $poly(F)$, where F is the number of columns of matrix $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$, *i.e.* the total number of freedoms allowed by the couplings in the mechanism.

3. When the mechanism analysed presents some degree of symmetry, the identical schemes enumerated can be successively eliminated by means of group theory, as proposed by Zhao et al.(18);

4. This method can be used to synthesise and optimise the actuation schemes of complex spatial mechanisms.

4.2.2 New algorithm for selecting optimal actuation scheme

In general for a given mechanism, different sets of feasible actuation scheme exist. When the complexity of the mechanism increases, the number of valid actuation schemes increases exponentially. Thus criteria for selecting an optimal actuation scheme among all the feasible ones enumerated are needed.

Based on the matroid formulation for a given mechanism presented in the previous section, a new method, original contribution of this Thesis, is proposed for the selection of actuation scheme.

Given a mechanism with g couplings allowing F single degree of freedoms, a weight w_i is attributed to each allowed degree of freedom i. The set of weights $\{w_i | w_i \in \mathcal{W}\}$ is chosen with respect to a set of criteria based on the mechanism specifications. In this way, each actuation scheme can be classified in terms of the sum of weights attributed to the corresponding active couplings.

The matrix $\hat{\boldsymbol{M}}_{\boldsymbol{N}}$ of the mechanism is now considered. Recalling that each column of $\hat{\boldsymbol{M}}_{\boldsymbol{N}}$ corresponds to a freedom allowed by the couplings of the mechanism, the set of weights $\{w_i | w_i \in \mathcal{W}\}$ can be considered assigned to the columns of $\hat{\boldsymbol{M}}_{\boldsymbol{N}}$.

The algebraic structure of matroids can now be conveniently employed in order to select actuation scheme. If each element of the ground set of E of a matroid $\mathcal{M}$ is given an arbitrary non-negative weight, the matroid optimisation problem consists in computing a basis with maximum total weight. This problem of determining the maximum weight independent set in a matroid can be solved using a greedy algorithm. Formal definitions and main theorem are presented in Appendix B.1.1.

The matroid $\mathcal{M}_{M_N}$ defined from matrix $\hat{\boldsymbol{M}}_{\boldsymbol{N}}$ of the given mechanism, as defined in the previous section, is considered. This matroid is thus a weighted matroid, with respect to which a greedy algorithm can be performed. The greedy algorithm for the dual matroids $\mathcal{M}^*_{M_N}$ returns the maximum weight basis for the matroid. This is exactly the maximum weight actuation set, among all feasible actuation schemes, which best fits the criteria specified for the mechanism.

A new algorithm for selecting an optimal actuation scheme of a given mechanism is herein proposed, based on the algebraic structure of matroid.

Algorithm 3 Selection of actuation scheme for a given mechanism

1: Create the network unit motion matrix $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$ for the given mechanism;
2: Create the linear matroid $\mathcal{M}_{M_N}$ on the real field $\mathbb{R}$ defined from matrix $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$;
3: Assign a weight to each freedom allowed by couplings in the mechanism, *i.e.* a weight is assigned to each column of $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$, based on a set of criteria (mechanism specifications);
4: Obtain the dual matroid $\mathcal{M}^*_{M_N}$;
5: Apply greedy algorithm to dual matroid $\mathcal{M}^*_{M_N}$;
6: The greedy algorithm returns the maximum weight independent set for $\mathcal{M}^*_{M_N}$;

Thus the set of variables corresponding to the maximum weight independent set is the actuation scheme which best fits the mechanism specifications. It suffices to notice that, while greedy algorithm gives in general a good approximation for a given problem, it is guaranteed to give the optimal solution if the problem is stated in matroid form (90). Alternatively the greedy algorithm characterises matroids, *i.e.* $\mathcal{M}$ is a matroid if and only if the greedy algorithm finds a basis B of maximum weight $w[B]$, for each weight function $w : W \to \mathbb{R}_+$. In Appendix B.1.1 this definition is stated formally.

The matroid formulation for a mechanism proposed in this Thesis allows thus to solve the problem of selecting an optimal actuation set with respect to a set of criteria established for the mechanism.

Algorithm complexity can now be calculated. As previously stated, for a spatial ($\lambda = 6$) mechanism with n links and g joints, matrix $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$ has size $6 \cdot (g - n + 1) \times 5 \cdot g$ in the worst case. The greedy algorithm has complexity $O(|E|f(r(\mathcal{M})) + |E|log|E|)$, as presented in Appendix B.1.1. The cardinality of ground set $|E|$ is $|E| = 5 \cdot g$. The rank of the dual matroid is $r(\mathcal{M}^*_{M_N}) = F_N$, where F_N is given by Equation (2.24).

The correctness of Algorithm (3) can be shown in the following form: Steps 1 implement the Davies' method, well established in literature: given a mechanism matrix $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$ can always be obtained (48, 17). Given a matrix $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$ a linear matroid $\mathcal{M}$ and its dual $\mathcal{M}^*$ can be defined (90) in Steps $2-4$. Greedy algorithm can be applied to any matroid (90) in Step $5-6$.

This algorithm has been implemented straightforwardly in the open- source mathematics software system SageMath (86). The greedy algorithm is already implemented in the software system as a method of the object *matroid()*.

Two examples of application of the this algorithm are presented in the next sections.

4.2.2.1 Selection of actuation set of *Tripteron* mechanism

The *Tripteron* mechanism, already introduced in Section 3.2.1.2, is considered for the analysis.

A slight structurally different of Tripteron is presented in Figure 37. A symmetric configuration is considered for analysis, anyway the results obtained in the following text are valid for any non-singular configuration of the mechanism. The parameters presented in Table 4 are considered for obtaining the characteristic matrices of the mechanism:

Table 4 – *Tripteron* mechanism parameters

l_2	l_3	l_6	l_7	l_8	l_9	d_{1x}	d_{5y}	d_{8y}	d_{8z}
1	1	1	1	1	1	1	1	2	1

where l_i is the length of link i and d_{ij} is the distance of link i from

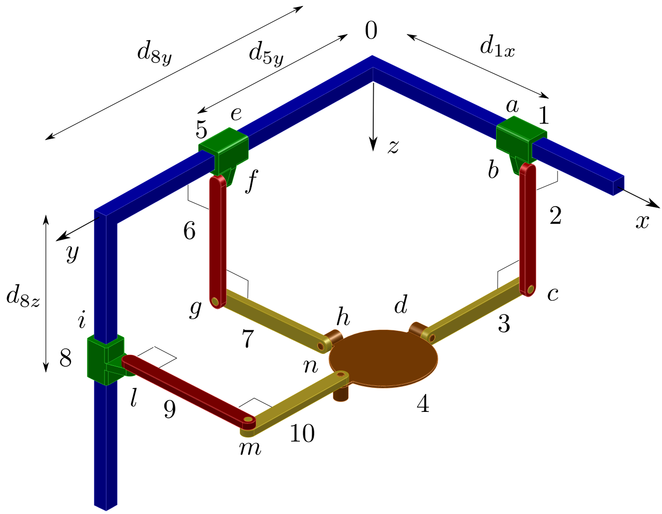


Figure 37 – Mechanism *Tripteron* with mobility $F_N = 3$ and $C_N = 3$.

origin along y-axis. Matrix $\left[\hat{\boldsymbol{M}}_{\boldsymbol{D}}\right]$ can thus be written as:

$$\left[\hat{\boldsymbol{M}}_{\boldsymbol{D}}\right] = \begin{array}{c} \begin{array}{cccccccccccc} \scriptstyle a & \scriptstyle b & \scriptstyle c & \scriptstyle d & \scriptstyle e & \scriptstyle f & \scriptstyle g & \scriptstyle h & \scriptstyle i & \scriptstyle l & \scriptstyle m & \scriptstyle n \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 2 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \tag{4.33}$$

The network unit motion matrix can now be obtained:

$$\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right] = \begin{array}{c} \begin{matrix} a & b & c & d & e & f & g & h & i & l & m & n \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 2 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \tag{4.34}$$

In order to apply the algorithm, a set of weights must be assigned to the columns $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$. Considering the topology of the mechanism, the following criterion for selecting actuation scheme is adopted:

- *The actuation set selected must contain the maximum number of linear actuators.*

Different criteria can be stated, corresponding to different weights sets. For the criterion considered, the set of weights $\{\mathcal{W}\}$ is assigned in the following form:

Table 5 – Columns weights for *Tripteron* mechanism

$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{d}$	$\boldsymbol{e}$	$\boldsymbol{f}$	$\boldsymbol{g}$	$\boldsymbol{h}$	$\boldsymbol{i}$	$\boldsymbol{l}$	$\boldsymbol{m}$	$\boldsymbol{n}$
5	2	3	4	5	2	3	4	5	2	3	4

Applying Algorithm (3) to the mechanism, the maximum independent set of the dual matroid $\mathcal{M}^*_{M_N}$ is obtained as:

$$\textit{Maximum independent set:}\ \ \{a, e, i\} \tag{4.35}$$

In this actuation scheme the three prismatic joints a, e and i are actuated. In fact, the mechanism *Tripteron* has been proposed with exactly this actuation scheme (70), thus the criterion adopted has been chosen for didactic purpose because of its simplicity. Different criteria can be stated in order to find an optimal basis. This actuation scheme correspond to a fully decoupled mechanism, *i.e.* each of the actuators is controlling one Cartesian degree of freedom, independently from the others (70).

An other criterion can be chosen. Suppose that the following criterion is adopted for selection of an actuation scheme:

- *The actuation set selected must prioritise: first, the maximum number of revolute coupling along y-axis, second, the maximum number of linear actuators and third, any other of the remaining coupling.*

whose corresponding set of weights $\{\mathcal{W}\}$ is assigned in the following form:

Table 6 – Columns weights for *Tripteron* mechanism - alternative criterion

$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{d}$	$\boldsymbol{e}$	$\boldsymbol{f}$	$\boldsymbol{g}$	$\boldsymbol{h}$	$\boldsymbol{i}$	$\boldsymbol{l}$	$\boldsymbol{m}$	$\boldsymbol{n}$
5	2	2	2	5	6	6	6	5	2	2	2

Applying Algorithm (3) to the mechanism, the maximum independent set of the dual matroid $\mathcal{M}^*_{M_N}$ is obtained as:

$$\textit{Maximum independent set:} \;\; \{e, g, h\} \tag{4.36}$$

where two revolute joints along y-axis are actuated (joints g and h) and the prismatic joint e is actuated. Regard that the set of the three revolute joint along y-axis, *i.e.* $\{f, g, h\}$ is not a valid set of actuation. In fact it can be verified that $\{f, g, h\}$ is not a basis for the dual matroid $\mathcal{M}^*_{M_N}$ or alternatively, the set $\{a, b, c, d, e, f, i, l, m, n\}$ is not a basis for the matroid $\mathcal{M}_{M_N}$. This means that the columns vectors $\{a, b, c, d, e, f, i, l, m, n\}$ of $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$ are linearly dependent.

A list of all feasible actuation schemes for the *Tripteron*, obtained applying the Algorithm (2) proposed in this Thesis, is included for completeness in Appendix C.4.

4.2.2.2 Selection of actuation set of *3-PPRR* mechanism

In general if some actuators are mounted close to the base of the robot, the total weight of the manipulator is decreased and the power-to-weight ratio is increased (91).

An optimal actuation set can be selected for the *3-PPRR* mechanism, depicted in Figure 35. The following criterion is employed to establish the coupling weights:

- *The actuation set selected must contain the actuators closer to basis* 0.

Matrices $\left[\hat{\boldsymbol{M}}_{\boldsymbol{D}}\right]$ and $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$ have been introduced respectively in Equations (C.19) and (C.20) presented in Appendix C.3.

Table 7 – Columns weights for *3-PPRR* mechanism

$\boldsymbol{b_1}$	$\boldsymbol{c_1}$	$\boldsymbol{d_1}$	$\boldsymbol{e_1}$	$\boldsymbol{b_2}$	$\boldsymbol{c_2}$	$\boldsymbol{d_2}$	$\boldsymbol{e_2}$	$\boldsymbol{b_3}$	$\boldsymbol{c_3}$	$\boldsymbol{d_3}$	$\boldsymbol{e_3}$
5	4	3	2	5	4	3	2	5	4	3	2

Applying Algorithm (3) to the mechanism, with the weights presented in Table 7, the maximum independent set of the dual matroid $\mathcal{M}^*_{M_N}$ is obtained as:

$$\textit{Maximum independent set:}\ \ \{b_1, b_2, b_3, c_i\} \quad \text{with } i = 1, 2, 3 \tag{4.37}$$

The actuation scheme comprising the three horizontal prismatic couplings $\{b_1, b_2, b_3\}$ and any one of the vertical prismatic coupling in the set $\{c_1, c_2, c_3\}$ is a maximum independent set satisfying the criterion.

4.3 SELF-ALIGNING MECHANISM ENUMERATION

In this section the matroid formulation is applied to the unit action matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$, in a similar way it has been applied to the unit motion matrix $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$.

Matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$ presents the following structures:

$$\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]_{\lambda k,C} = \begin{array}{c} \begin{array}{cccc} \overbrace{* \;\; * \;\; \dots}^{\text{Coupling a}} & \overbrace{* \;\; \dots}^{\text{Coupling b}} & \overbrace{\;\dots\;}^{\dots} & \overbrace{\dots \;\; *}^{\dots} \end{array} \\ \left[\begin{array}{ccc:cc:c:cc} * & * & \dots & * & \dots & \dots & \dots & * \\ * & * & \dots & * & \dots & \dots & \dots & * \\ & & & & & & & \\ * & * & \dots & * & \dots & \dots & \dots & * \\ * & * & \dots & * & \dots & \dots & \dots & * \\ * & * & \dots & * & \dots & \dots & \dots & * \\ & & & & & & & \\ * & * & \dots & * & \dots & \dots & \dots & * \\ & & & & & & & \\ * & * & \dots & * & \dots & \dots & \dots & * \\ * & * & \dots & * & \dots & \dots & \dots & * \\ & & & & & & & \\ * & * & \dots & * & \dots & \dots & \dots & * \end{array}\right] \end{array} \begin{array}{l} \left.\begin{array}{l} \}\ \text{dim. } 1 \\ \}\ \text{dim. } 2 \\ \qquad\vdots \\ \}\ \text{dim. } \lambda \end{array}\right\}\text{Cutset } 1 \\ \left.\begin{array}{l} \}\ \text{dim. } 1 \\ \}\ \text{dim. } 2 \\ \qquad\vdots \\ \}\ \text{dim. } \lambda \end{array}\right\}\text{Cutset } 2 \\ \qquad\qquad\qquad\vdots \\ \left.\begin{array}{l} \}\ \text{dim. } 1 \\ \}\ \text{dim. } 2 \\ \qquad\vdots \\ \}\ \text{dim. } \lambda \end{array}\right\}\text{Cutset } k \end{array} \tag{4.38}$$

where each column is uniquely determined by a single constraint imposed by a coupling. Dependence and independence of the mechanism's constraints can thus be analysed by studying the properties of $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$. As introduced in Section 3.3.1, the independent columns of $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$ identify a set of leading variables of the linear homogeneous system (3.41), *i.e.* a set of independent constraints of the mechanism. Each set of independent constraints form a basis for the constraint space of the mechanism, and can be regarded as a mechanism free of redundant constraints, *i.e.* a self-aligning mechanism, derived from the original one. Thus given a mechanism with C constraints and C_N redundant ones, a self-aligning mechanism kinematically equivalent to the original mechanism can be derived removing a set of C_N redundant constraints. The self-aligning mechanism derived thus has $C' = C - C_N$ constraints, all independent ones, and the linear homogeneous system (3.41) can be written as:

$$\left[\hat{\boldsymbol{A}}'_{\boldsymbol{N}}\right]_{\lambda k, C-C_N} [\boldsymbol{\Psi}]_{C-C_N,1} = [\boldsymbol{0}]_{\lambda k} \tag{4.39}$$

where $\left[\hat{\boldsymbol{A}}'_{\boldsymbol{N}}\right]_{\lambda k, C-C_N}$ has full rank $rank(\left[\hat{\boldsymbol{A}}'_{\boldsymbol{N}}\right]_{\lambda k, C-C_N}) = C - C_N$. In the same way to the Condition (2), a new condition is herein stated for a self-aligning mechanism:

Condition 3. *For a mechanism with C constraints and C_N redundant ones in a given configuration, a set of $C' = C - C_N$ constraints represents a self-aligning mechanism kinematically equivalent to the original one, if and only if, in the given configuration, the corresponding matrix $\left[\hat{\boldsymbol{A}}'_{\boldsymbol{N}}\right]_{\lambda k, C-C_N}$, obtained from $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]_{\lambda k, C}$ removing the C_N columns corresponding to the set of redundant constraint, has full rank $rank(\left[\hat{\boldsymbol{A}}'_{\boldsymbol{N}}\right]_{\lambda k, C-C_N}) = C - C_N$.*

In general different basis can be chosen to span the constraint space, thus different self-aligning mechanism can be derived. All combination of $C - C_N$ columns of $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]_{\lambda k, C}$ can be tested for linear independence in order to find a basis for the constraint space (85). This method is computationally heavily time-consuming because of the combinatorial explosion.

A new approach, which is an original contribution of this Thesis, is herein proposed. Following the same approach used in the previous section for analysing actuation schemes, matroid theory is introduced.

Given a mechanism and its associated network unit action matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]_{\lambda k, C}$, a linear matroid $\mathcal{M}_{A_N}$ is defined over the reals $\mathbb{R}$ from matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$, where the ground set $E(\mathcal{M}_{A_N})$ denotes the index set of the columns of $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]_{\lambda K, C}$ $E = \{1, 2, \ldots, C\}$.

For a subset X of E, let $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]_{\boldsymbol{X}}$ denote the submatrix of $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$ consisting of those columns indexed by X. A family $\mathcal{I}$ of subset can now be define for the matroid $\mathcal{M}$:

$$\mathcal{I} = \{X \subseteq E : \text{rank } ([A_N]_X) = |X|\} \tag{4.40}$$

where a set X is independent if the corresponding columns are linearly independent. A basis B corresponds to a linearly independent set of columns of cardinality rank $(\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]) = C - C_N$.

Thus the column space of $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$ is spanned any basis B belonging to the collection of bases of the matroid $\mathcal{B}(\mathcal{M}_{A_N})$. Regarding that a basis B corresponds to a maximal set of independent columns of $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$, any basis B of the matroid $\mathcal{M}_{A_N}$ represents a self-aligning mechanism kinematically equivalent to the original one.

The problem of enumerating all self-aligning mechanisms of a given mechanism has thus been turned into the problem of enumerating all the bases of matroid $\mathcal{M}_{A_N}$, which is a well-known problem in matroid theory. In next section a new algorithm, based on matroid bases enumeration, is proposed.

4.3.1 New algorithm for enumeration of all self-aligning mechanism derived from a given one.

A new algorithm for enumeration of all self-aligning mechanism kinematically equivalent to a given mechanism is proposed.

Algorithm 4 Enumeration of all self-aligning mechanisms kinematically equivalent to a given mechanism

1: Create the network unit motion matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$;
2: Create the linear matroid $\mathcal{M}_{A_N}$ on the real field $\mathbb{R}$ defined over matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$;
3: Enumerate all bases B of matroid $\mathcal{M}_{A_N}$;

Thus the sets of variables corresponding to the bases of matroid represent the self-aligning mechanisms derived form the given mechanism. As for the previous Algorithms (2) and (3), this algorithm has been implemented straightforwardly in the open-source Sagemath software (86).

In order to analyse the complexity of Algorithm (4), the worst case in terms of matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$ size is analysed. As stated in Section 3.4.1, for a spatial ($\lambda = 6$) mechanism with n links and g joints, matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$ has size $6 \cdot (n-1) \times 5 \cdot g$ in the worst case. Thus, recalling Equation (B.7), each base of the matroid is enumerated in polynomial time $poly(5 \cdot g)$ (88), (89).

The cardinality of each base is the number of independent constraints, *i.e.* $C - C_N$ of the mechanism. Thus, in the worst case the number of $\mathcal{M}_{A_N}$ bases, *i.e.* the cardinality of $\mathcal{B}(\mathcal{M}_{A_N})$, is $(C - C_N)$-combination of the ground set E:

$$|\mathcal{B}^*(\mathcal{M}_{M_N})| = \left(\begin{array}{c} 5 \cdot g \\ C - C_N \end{array} \right) \tag{4.41}$$

The correctness of Algorithm (2) can be shown in the following form: Steps 1 implement the Davies' method, well established in literature: given a mechanism, matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$ can always be obtained (48, 17). Given a matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$ a linear matroid $\mathcal{M}_{AN}$ can be defined (90) in Steps 2. In Step 3 all the bases of matroid $\mathcal{M}_{A_N}$ can be enumerated (89).

In the next section two example of application of the algorithm are proposed. First, the actuation of a simple planar mechanism is analysed and all actuation schemes are enumerated. Then the complex spatial mechanism *Tripteron*, already considered in the previous sections, is analysed.

4.3.1.1 Example: enumeration of all self-aligning mechanism derived from a planar overconstrained mechanism.

Algorithm (4) is applied to the planar mechanism in Figure 38, with five links and six revolute couplings a, b, c, d, e and f. The mechanism has mobility $F_N = 1$ and $C_N = 1$ redundant restriction.

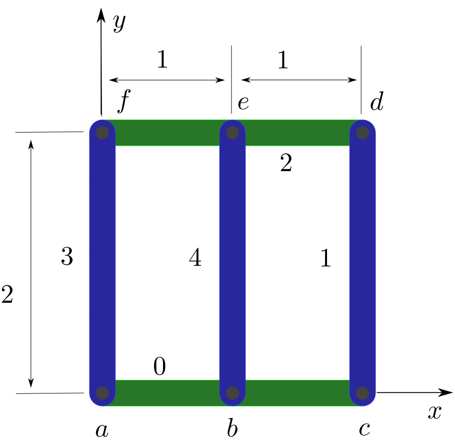


Figure 38 – Planar mechanism with mobility $F_N = 1$ and redundant constraint $C_N = 1$

As the mechanism is planar, only the planar components of planar motion are considered, *i.e.* t, u and v. The motion screws of the coupling network thus belong to the fifth special 3-system of screws

(2). The unit motion matrix can be written as:

$$\left[\hat{\boldsymbol{M}}_{\boldsymbol{D}}\right] = \begin{array}{c c} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{array}{c} t \\ u \\ v \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & -1 & -2 & -2 & -1 & 0 \end{bmatrix} \end{array} \tag{4.42}$$

In the same way, only the planar components of constraints are considered, *i.e.* T, U and V. The screw system of the circuit actions caused by overconstraint belong to the fourth special 3-system of screws. For the planar case, with the order of the screw system to which all screws belong is $\lambda = 3$, each revolute coupling can transmit two actions: U and V. The unit action matrix can be written as:

$$\left[\hat{\boldsymbol{A}}_{\boldsymbol{D}}\right]_{3,12} = \begin{array}{c c} & \begin{array}{cccccccccccc} a_U & a_V & b_U & b_V & c_U & c_V & d_U & d_V & e_U & e_V & f_U & f_V \end{array} \\ \begin{array}{c} T \\ U \\ V \end{array} & \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 2 & -2 & 2 & -2 & 1 & -2 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array} \tag{4.43}$$

Based on the graph G_A of the mechanism, presented in Figure 39, the cutset matrix Q is:

$$[\boldsymbol{Q}]_{4,6} = \begin{array}{c c} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{array}{c} b \\ d \\ e \\ f \end{array} & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \tag{4.44}$$

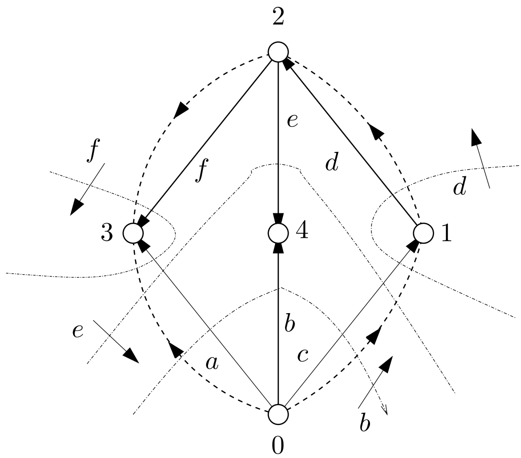


Figure 39 – Action graph G_A associated with mechanism of Figure 38.

The network unit action matrix can now be obtained as:

$$\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]_{12,12} = \begin{array}{c} \begin{array}{cccccccccccc} a_U & a_V & b_U & b_V & c_U & c_V & d_U & d_V & e_U & e_V & f_U & f_V \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \tag{4.45}$$

A linear matroid $\mathcal{M}_{A_N}$ over the real number $\mathbb{R}$ is defined over matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$. The set of bases of the matroid are enumerated. Each basis correspond to a self-aligning mechanism kinematically equivalent to the one presented in Figure 38.

The set of bases $\mathcal{B}$ of $\mathcal{M}_{A_N}$ is:

$$\mathcal{B}(\mathcal{M}_{A_N}) = \{a_U, b_U, b_V, c_U, c_V, d_U, d_V, e_U, e_V, f_U, f_V\} \tag{4.46}$$
$$\{a_U, a_V, b_U, c_U, c_V, d_U, d_V, e_U, e_V, f_U, f_V\} \tag{4.47}$$
$$\{a_U, a_V, b_U, b_V, c_U, d_U, d_V, e_U, e_V, f_U, f_V\} \tag{4.48}$$
$$\{a_U, a_V, b_U, b_V, c_U, c_V, d_U, e_U, e_V, f_U, f_V\} \tag{4.49}$$
$$\{a_U, a_V, b_U, b_V, c_U, c_V, d_U, d_V, e_U, f_U, f_V\} \tag{4.50}$$
$$\{a_U, a_V, b_U, b_V, c_U, c_V, d_U, d_V, e_U, e_V, f_U, \} \tag{4.51}$$

It can be observed that links 1, 3 and 4 must have the same length in order to assemble the mechanism with one degree of freedom, because of the presence of one redundant constraint. Moreover, in the given mechanism's configuration, the translation along y-axis is exactly the freedom which is overconstrained.

Examining the result of the proposed algorithm, each one of the corresponding self-aligning mechanism obtained has exactly one constrained along y-axis removed from a coupling, more precisely:

1. Mechanism (4.46) has the constraint along y-axis removed from coupling a, *i.e.* the constraint a_V has been removed;

2. Mechanism (4.47) has the constraint along y-axis removed from coupling b, *i.e.* the constraint b_V Has been removed;

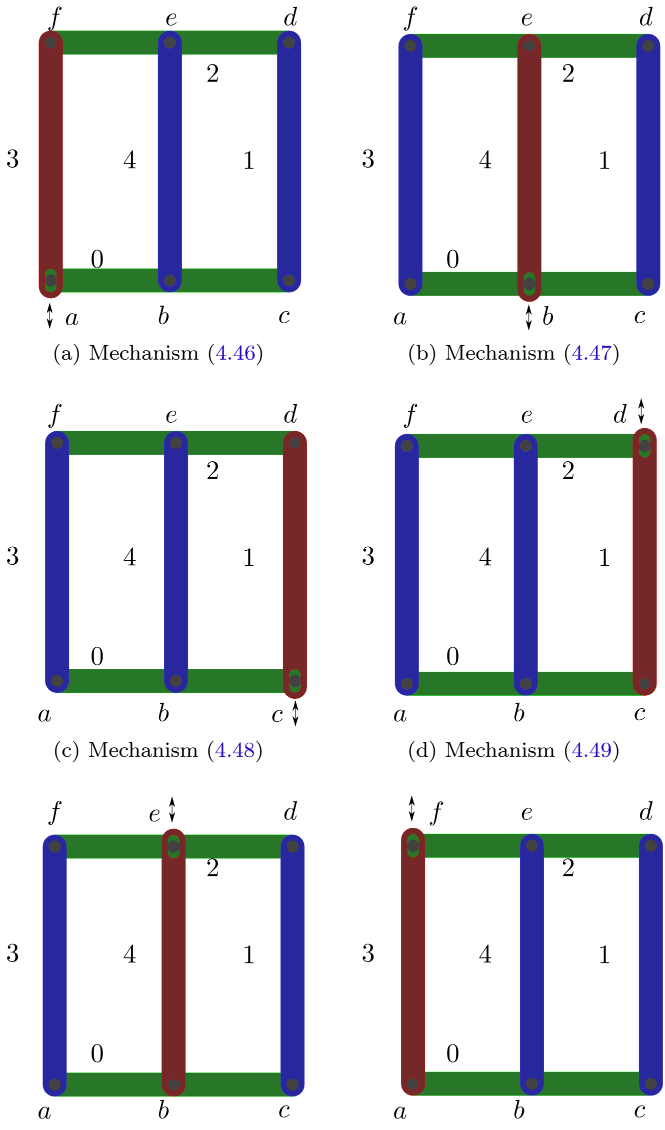
3. Mechanism (4.48) has the constraint along y-axis removed from coupling c, *i.e.* the constraint c_V has been removed;

4. Mechanism (4.49) has the constraint along y-axis removed from coupling d, *i.e.* the constraint d_V has been removed;

5. Mechanism (4.50) has the constraint along y-axis removed from coupling e, *i.e.* the constraint e_V has been removed;

6. Mechanism (4.51) has the constraint along y-axis removed from coupling f, *i.e.* the constraint f_V has been removed.

The self-aligning mechanism enumerate by the algorithm are presented in Figure 40.



(a) Mechanism (4.46)
(b) Mechanism (4.47)
(c) Mechanism (4.48)
(d) Mechanism (4.49)
(e) Mechanism (4.50)
(f) Mechanism (4.51)

Figure 40 – The self-aligning mechanisms kinematically equivalent to the overconstrained mechanism shown in Figure 38.

4.3.1.2 Example: enumeration of self-aligning mechanisms for a spatial overconstrained mechanism.

Algorithm (4) can be applied to the *Tripteron* mechanism, already analysed in Section 3.2.1.2 and 4.2.2.1. The network unit action matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$ is presented in Appendix C.4, based on the configuration depicted in Figure 37.

The mechanism considered presents a total of $C = 60$ constraints, of which $C_N = 3$ are redundant constraints.

A linear matroid $\mathcal{M}_{A_N}$ over the real number $\mathbb{R}$ is defined over matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$. The set of bases of the matroid can thus be enumerated. Each basis correspond to a self-aligning mechanism kinematically equivalent to the one presented in Figure 37.

The Algorithm (4) enumerates a total of 512 distinct basis for the matroid $\mathcal{M}_{A_N}$, *i.e.* the mechanism *Tripteron* presents 512 self-aligning distinct mechanisms, all of them kinematically equivalent to the *Tripteron* mechanism.

It is important to regard that, as the complexity of the mechanism increases, the numbers of possible self-aligninging mechanisms grows exponentially. It is thus important a way of selecting the best self-aligning mechanism with respect to some criteria. In the following section, a new algorithm, original contribution of this Thesis, is proposed for selecting a self-aligning mechanism which best fits a criterion based on mechanism specifications.

4.4 CLASSIFICATION OF SELF-ALIGNING MECHANISM

In general for a given mechanism distinct self-aligning mechanisms kinematically equivalent exist. When the complexity of the mechanism increases, the numbers of possible self-aligning mechanisms grows exponentially. Thus criteria for selecting an optimal self-aligning mechanism between all possible ones are needed.

Based on the matroid formulation presented in the previous sections, a new method, original contribution of this Thesis, is proposed for selection of self-aligning mechanism.

Given a mechanism with g couplings imposing C constraints, a weight w_i is attributed to each allowed constraint c_i. The set of weights $\{w_i | w_i \in \mathcal{W}\}$ is chosen with respect to a set of criteria based on the mechanism specifications. In a similar way of Algorithm (3), each self-aligning mechanism can be classified in terms of the sum of weights attributed to the corresponding active couplings.

The matrix $\hat{\boldsymbol{A}}_{\boldsymbol{N}}$ of the mechanism is considered. Recalling that each column of $\hat{\boldsymbol{A}}_{\boldsymbol{N}}$ corresponds to a constraint imposed by the couplings of the mechanism, the set of weights $\{w_i | w_i \in \mathcal{W}\}$ can be considered assigned to the columns of $\hat{\boldsymbol{A}}_{\boldsymbol{N}}$.

Thus the problem of selecting an optimal self-aligning mechanism is transformed in the problem of determining the maximum weight independent set in a matroid. As already stated previously, this problem can be solved using a greedy algorithm in polynomial time.

The matroid $\mathcal{M}_{A_N}$ defined from matrix $\hat{\boldsymbol{A}}_{\boldsymbol{N}}$ of the given mechanism, as defined in the previous section, is considered. This matroid is thus a weighted matroid, with respect to which a greedy algorithm can be performed. The greedy algorithm for the dual matroids $\mathcal{M}^*_{A_N}$ returns the maximum weight basis for the matroid. This is exactly the maximum weight self-aligning mechanism, among all possible self-aligning mechanism kinematically equivalent, which best fits the criteria specified for the mechanism.

A new algorithm for selecting an optimal self-aligning mechanism is herein proposed, based on the algebraic structure of matroid.

Algorithm 5 Selection of self-aligning mechanism, kinematically equivalent to a given mechanism

---

1: Create the network unit action matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$ for the given mechanism;
2: Create the linear matroid $\mathcal{M}_{A_N}$ on the real field $\mathbb{R}$ defined from matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$;
3: Assign a weight to each constraint imposed by couplings in the mechanism, *i.e.* a weight is assigned to each column of $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$, based on a set of criteria (mechanism specifications);
4: Apply greedy algorithm to matroid $\mathcal{M}_{A_N}$;
5: The greedy algorithm returns the maximum weight independent set for $\mathcal{M}_{A_N}$;

Thus the set of variables corresponding to this set is the self-aligning mechanism which best fits the mechanism specifications. The matroid structure guarantees that the solution encountered is the optimal one. The matroid formulation for a mechanism proposed in this Thesis allows thus to solve the problem of finding an optimal self-aligning mechanism with respect to a set of criteria established for the mechanism.

Algorithm complexity can now be calculated. For a spatial ($\lambda = 6$) mechanism with n links and g joints, matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$ has size $6 \cdot (n-1) \times 5 \cdot g$ in the worst case. The greedy algorithm has complexity $O(|E|f(r(\mathcal{M})) + |E|log|E|)$, as presented in Appendix B.1.1. The cardinality of ground set $|E|$ is $|E| = 5 \cdot g$. The rank of the matroid is $r(\mathcal{M}_{A_N}) = C - C_N$.

The correctness of Algorithm (3) can be shown in the following form: Steps 1 implement the Davies' method, well established in literature: given a mechanism matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$ can always be obtained (48, 17). Given a matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$ a linear matroid $\mathcal{M}_{A_N}$ can be defined (90) in Steps 2. Greedy algorithm can be applied to any matroid (90) in Step $3-5$.

This algorithm has been implemented straightforwardly in the open- source mathematics software system SageMath (86). The greedy algorithm is already implemented in the software system as a method of the object *matroid()*.

4.4.0.3 Example: selection of an optimal self-aligning mechanism for spatial overconstrained mechanism.

The Algorithm (5) can be applied to the *Tripteron* mechanism. As presented in Section 4.3.1.2, the *Tripteron* mechanism presents 512 possible self-aligning mechanism, all kinematically equivalent to the *Tripteron*.

In order to apply the algorithm, a set weights must be assigned to the columns $\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right]$. Considering the topology of the mechanism, the following criterion for selecting actuation scheme is adopted:

- *The self-aligning mechanism must contain the three prismatic couplings, which will be actuated. Furthermore revolute couplings in the set* $\{g, h, m, n, c, d\}$ *should be maintained. Furthermore any planar freedom in the set* $\{b, l, f\}$ *should be avoided,* i.e. *the planar constraints* U*,* V *and* T *should be maintained in joints* $\{b, l, f\}$.

This criterion is mostly for didactic purpose, and it aims to obtain a self-aligning mechanism easy to control and with standard joints (revolute, universal and spherical), avoiding more complicated joints as the planar ones. Different criteria can be stated, corresponding to different weights sets. For the criterion considered, the set of weights $\{\mathcal{W}\}$ is assigned in Table 8.

Table 8 – Constraints for *Tripteron* mechanism

Contr.	*W*	*Contr.*	*W*	*Contr.*	*W*	*Contr.*	*W*
$\boldsymbol{a_R}$	5	$\boldsymbol{d_S}$	5	$\boldsymbol{g_R}$	5	$\boldsymbol{l_R}$	0
$\boldsymbol{a_S}$	5	$\boldsymbol{d_T}$	5	$\boldsymbol{g_T}$	5	$\boldsymbol{l_S}$	0
$\boldsymbol{a_T}$	5	$\boldsymbol{d_U}$	5	$\boldsymbol{g_U}$	5	$\boldsymbol{l_U}$	2
$\boldsymbol{a_V}$	5	$\boldsymbol{d_V}$	5	$\boldsymbol{g_V}$	5	$\boldsymbol{l_V}$	2
$\boldsymbol{a_W}$	5	$\boldsymbol{d_W}$	5	$\boldsymbol{g_W}$	5	$\boldsymbol{l_W}$	2
$\boldsymbol{b_S}$	0	$\boldsymbol{e_R}$	5	$\boldsymbol{h_R}$	5	$\boldsymbol{m_R}$	5
$\boldsymbol{b_T}$	0	$\boldsymbol{e_S}$	5	$\boldsymbol{h_T}$	5	$\boldsymbol{m_S}$	5
$\boldsymbol{b_U}$	2	$\boldsymbol{e_T}$	5	$\boldsymbol{h_T}$	5	$\boldsymbol{m_U}$	5
$\boldsymbol{b_V}$	2	$\boldsymbol{e_U}$	5	$\boldsymbol{h_V}$	5	$\boldsymbol{m_V}$	5
$\boldsymbol{b_W}$	2	$\boldsymbol{e_W}$	5	$\boldsymbol{h_W}$	5	$\boldsymbol{m_W}$	5
$\boldsymbol{c_S}$	5	$\boldsymbol{f_R}$	0	$\boldsymbol{i_R}$	5	$\boldsymbol{n_R}$	5
$\boldsymbol{c_T}$	5	$\boldsymbol{f_T}$	0	$\boldsymbol{i_S}$	5	$\boldsymbol{n_S}$	5
$\boldsymbol{c_U}$	5	$\boldsymbol{f_U}$	2	$\boldsymbol{i_T}$	5	$\boldsymbol{n_U}$	5
$\boldsymbol{c_V}$	5	$\boldsymbol{f_V}$	2	$\boldsymbol{i_U}$	5	$\boldsymbol{n_V}$	5
$\boldsymbol{c_W}$	5	$\boldsymbol{f_W}$	2	$\boldsymbol{i_V}$	5	$\boldsymbol{n_W}$	5

With the criterion considered the algorithm proposed looks for a self-aligning mechanism by eliminating a set of planar constraints in couplings $\{b, f, l\}$.

Applying Algorithm (3) to the mechanism, the maximum independent set of the matroid $\mathcal{M}_{A_N}$ is obtained as:

$$\begin{aligned} \textit{Max. independent set: } \{ & a_R, a_S, a_T, a_V, a_W, b_T, b_U, b_V, b_W, c_S, c_T, c_U, \\ & c_V, c_W, d_S, d_T, d_U, d_V, d_W, e_R, e_S, e_T, e_U, e_W, \\ & f_U, f_V, f_W, g_R, g_T, g_U, g_V, g_W, h_R, h_T, h_U, h_V, \\ & h_W, i_R, i_S, i_T, i_U, i_V, l_R, l_S, l_U, l_V, l_W, m_R, m_S, \\ & m_U, m_V, m_W, n_R, n_S, n_U, n_V, n_W\} \end{aligned} \tag{4.52}$$

The Set (4.52) describes a self-aligning mechanism kinematically equivalent to the *Tripteron*. This set describes the constraints which are maintained for each joint. For example for coupling a five constraints are maintained, more exactly R,S,T,V and W, thus a is a prismatic pair along x axis. On the other hand, for joint B four constraints are maintained, more exactly T,U,,V and W, thus b is an universal pair with rotations along x and y axis. In the same way all remaining

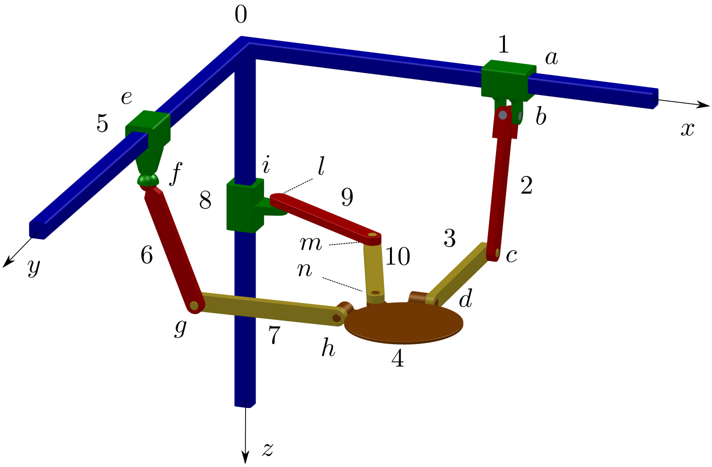


Figure 41 – Self-aligning mechanism derived from Tripteron.

joints can be obtained from Set (4.52). The corresponding self-aligning mechanism is presented in Figure 41.

It is important to notice that the greedy algorithm is guaranteed to find an optimal solution when performing on matroid structure. In general this solution is not unique. Thus Algorithm (5) proposed in this Thesis returns one possible self-aligning mechanism which satisfies the given criteria. All self-aligning mechanism satisfying the given criteria can be enumerated by calculating the weight of each basis enumerated by Algorithm (4) and selecting the maximal ones. Establishing the criteria in terms of the given specifications for a mechanism is thus of the greatest importance for the enumeration of interesting sets of self-aligning mechanisms. This problem of establishing valid criteria in terms of mechanism specifications is addressed as future work of this Thesis.

5 COMPLEMENTARY RESULTS

In this section two complementary results, original contributions of this thesis, are also introduced. First a new kinematic chain invariant is introduced, which states a relation between mobility and degree of constraints for a given mechanism. Then a counterexample for the method of topological analysis for overconstrained mechanism proposed by Reshetov(26), (92) is introduced. The method proposed by Reshetov(26) is analysed by means of screw theory and a flaw in the method is identified.

5.1 A NEW KINEMATIC CHAIN INVARIANT

In this section, a new kinematic chain invariant is introduced. This result is an original contribution of this Thesis. Consider the kinematic chain presented in Figure 42a.

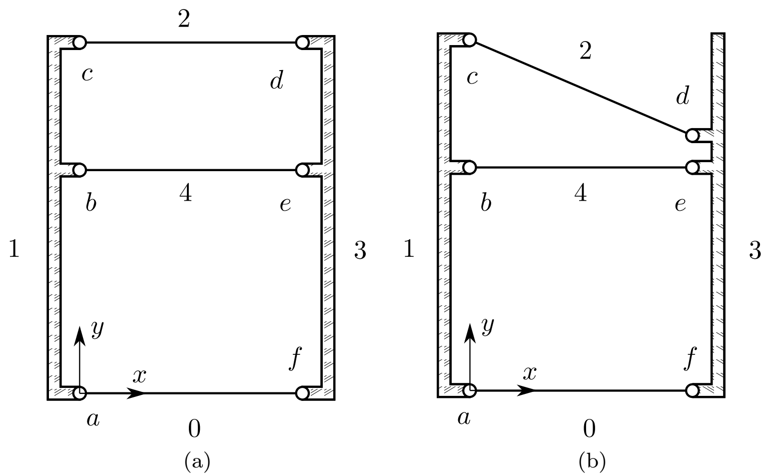


Figure 42 – Two mechanisms with the same number of links and joints, $n = 5$ links and $g = 6$ joints, with $F_N = 1$ and $C_N = 1$ in (a) and $F_N = 0$ and $C_N = 0$ in (b)

The mechanism has $n = 5$ links and $g = 6$ joints. By visual inspection a redundant constraint can be identified in the mechanism along x-axis. In Appendix C.5 Davies' equation is applied to this mechanism in order to calculate the number of redundant constraints C_N.

The same mechanism has been analysed in Section 4.3.1.1 in order to find all self-aligning mechanisms kinematically equivalent. Recalling the Modified Grübler-Kutzbach Criterion in the form:

$$F_N = \lambda(n - g - 1) + \sum_{i=1}^{g} f_i + C_N \tag{5.1}$$

Equation (5.1) can can be rearranged as:

$$F_N = \mathcal{F} + C_N \tag{5.2}$$

where mobility can be expressed as the sum of components $\mathcal{F}$ and C_N. The component $\mathcal{F}$ can be written as:

$$\mathcal{F} = f\left(\lambda, n, g, f_i\right) = \lambda(n - g - 1) + \sum_{i=1}^{g} f_i \tag{5.3}$$

and depends exclusively on the topology of the mechanism, *i.e.* on the number of links and number and type of joints, and it can be regarded as an invariant of the kinematic chain. The invariant $\mathcal{F}$ depends also on the order of the screw system λ, which is the minimum order of the screw system to which all motion and action screws belong. Alternatively, in the most general case, it can be considered $\lambda = 6$.

The component C_N, the number of redundant constraints, depend on the structure of the mechanism, *i.e.* on the number and on the position of links and joints. Equation (5.2) can be thus rearranged as:

$$F_N - C_N = \mathcal{F} \tag{5.4}$$

which states that, given a kinematic chain with n links and g joints, the difference between the mobility F_N and the number of redundant constraint C_N is a constant, and it is exactly the invariant $\mathcal{F}$ defined above. Equation (5.4) means that for a given kinematic chain, if the position or orientation of joints is modified such that the degree of overconstraint C_N varies, than the mobility of the mechanism F_N varies too, in order to keep the invariant $\mathcal{F}$ constant.

For the mechanism of Figure 42a $\lambda = 3$ as the mechanism is planar, thus $\mathcal{F} = 3 \cdot (5 - 6 - 1) + 6 = 0$ and Equation 5.4 is holds as $1 - 1 = 0$.

Now the mechanism in Figure 42b is considered. This mechanism has the same number of links and joints as the one depicted in Figure 42a, but the position of joint d has been changed. Therefore, as the type and number of joints did not change, the invariant do not change for the mechanism of Figure 42b, *i.e.* $\mathcal{F} = 0$ as expected. Thus any change in C_N implies a equal change in F_N, in order to keep the invariant $\mathcal{F}$ constant.

Applying Davies' equations, as presented in Appendix C.5, it can be verified that no redundant constraints are present in the mechanism, *i.e.* $C_N = 0$, and thus the mobility is $F_N = 0$ by Equation (5.4). Thus mechanism of Figure 42b is a truss.

Another example is considered. The planar kinematic chain presented in Figure 43 is a truss, *i.e.* a mechanism with mobility $F_N = 0$ and redundant constraints $C_N = 0$. The number of links is $n = 15$ and the number of joints is $g = 21$.

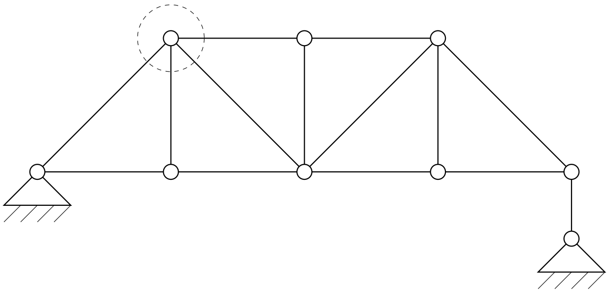
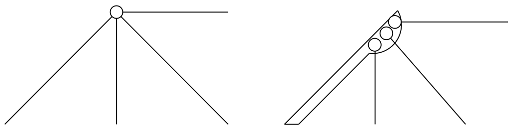


Figure 43 – Planar kinematic chain with $n = 15$ links and $g = 21$ joints, $M = 0$, $C_N = 0$ and $\mathcal{F} = 0$.

Assume that the joints in Figure 43 are multiple joints. In Figure 44 a multiple joint (circled in Figure 43) is expanded for a correct evaluation of the number of joints. Thus the multiple joint of Figure 44a is composed by three single joints as in Figure 44b.



(a) Mechanism (4.46) (b) Mechanism (4.47)

Figure 44 – A multiple joint (a) and its expansion (b).

Applying Equation (5.3), the invariant for this truss is $\mathcal{F} = 3(15 - 21 - 1) + 21 = 0$. In Figure 45 a kinematic chain with the same number of link and joints is presented, but with one member (represented as a dashed line) reallocated. As in the previous example, the type and number of joints did not change and the kinematic chain presents the same invariant $\mathcal{F} = 0$, considering the mechanism planar with $\lambda = 3$.

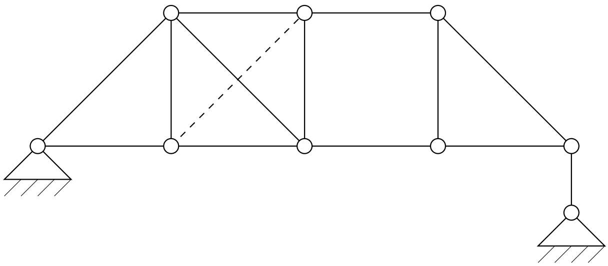


Figure 45 – Planar kinematic chain with $n = 15$ links and $g = 21$ joints, $M = 1$ and $C_N = 1$, but still $\mathcal{F} = 0$.

In this kinematic chain the number of redundant constraints is $C_N = 1$, thus the mobility must change in order to keep the invariant $\mathcal{F}$ constant. By Equation (5.2), the mobility is thus $F_N = 1$.

Therefore, for a given kinematic chain, if the position or orientation of joints is modified such that the degree of overconstraint C_N (mobility F_N) varies, than the mobility (degree of overconstraint C_N) of the mechanism F_N varies too, in order to keep the invariant $\mathcal{F}$ constant.

5.2 DISCUSSION ABOUT RESHETOV METHOD

In this section the original method proposed by Reshetov (26), (92) for analysis redundant constraints of a given mechanism is reviewed. Then new proofs for propositions stated by Reshetov(26) are introduced in terms of screw theory, which are an original contribution of this Thesis. A new counterexample for this method is introduced and analysed by means of screw theory.

5.2.1 Review of Reshetov method

The method proposed by Reshetov (26), (92) focus on the analysis and detection of redundant constraints in a given mechanism. The method is based on a simple observation that all mechanisms (with the exception of open loop mechanisms like manipulators) are made of loops of links connected by joints. If a mechanism is designed without redundant constraints, then even if the links are deformed, it should be possible to close each loop without use of force.

The number of independent loops ν of a given mechanism can be calculated as:

$$\nu = g - n + 1 \tag{5.5}$$

where n is the number of links and g is the number of joints in the mechanism. Malytsheff(93) introduced the structural equation:

$$F_N = 6 \cdot n - (\sum_{i=1}^{i=5} i \cdot p_i - C_N) \tag{5.6}$$

where p_i is the number of kinematic pairs of class i, where each kinematic pair of class i imposes i constraints. For example a revolute coupling belonging to class V imposes 5 constraints. On the other hand, a spherical pair belonging to class III imposes 3 constraints. Thus $\sum_{i=1}^{i=5} i \cdot p_i$ represents the sum of all constraints applied to the mechanism by its couplings. It is important to regard that Equation 5.6 is referred to as the unified equation for mobility mechanism (62), (13).

Equation (5.6) can be expanded in the form:

$$C_N = F_N - 6 \cdot n + 5 \cdot p_V + 4 \cdot p_{IV} + 3 \cdot p_{III} + 2 \cdot p_{II} + p_I \tag{5.7}$$

An analogous formulation of Equation (5.6) has been suggested by Ozol(94):

$$F_N = F - 6 \cdot \nu + C_N. \tag{5.8}$$

where F is the sum of all the degree of freedom allowed by the joints in the mechanism. Thus $F = \sum_{i=1}^{j} f_i$, where f_i is the freedom allowed by joint j. The sum F can be decomposed along the coordinate axes in the form:

$$F = f_{tx} + f_{ty} + f_{tz} + f_{rx} + f_{ry} + f_{rz} \tag{5.9}$$

where f_{tx}, f_{ty} and f_{tz} are the sum of liner mobilities along respectively x, y and z-axes,e f_{rx}, f_{ry} e f_{rz} are the sum of angular mobilities along the same axes.

Based on Equation (5.9) the following proposition is stated by Reshetov(26):

Proposition 1. *For a single-loop mechanism, the presence of all the three angular mobilities is a necessary condition for the loop to close without strain,* i.e. $f_{rx} \geq 1$, $f_{ry} \geq 1$ *e* $f_{rz} \geq 1$.

Thus the absence of a single angular mobility implies the presence of strain in the mechanism and thus redundant constraints. On the other hand, a similar condition is not necessary for linear mobilities, if extra angular mobilities are present in the mechanism. In fact a linear mobility can be obtained not only as a linear displacement of some links, but also rotating the links about an axis perpendicular to the direction of linear mobility. The following proposition is stated by Reshetov(26):

Proposition 2. *A linear mobility can be replaced by angular mobility about an axis perpendicular to the direction of linear mobility.*

However the mechanism must be checked to verify if a link would move linearly in the required direction as a result of such rotation, and if this capability is still valid in all configurations of the mechanism. An example is presented for a better understanding.

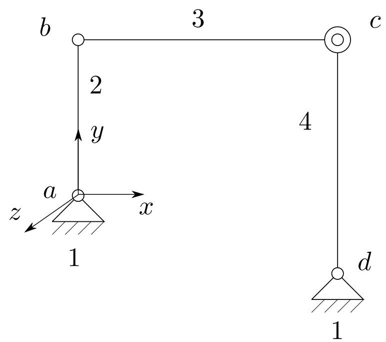


Figure 46 – Spatial four-bar mechanism with 3 revolute couplings a, b, d and a spherical one c

In Figure 46 a spatial four-bar mechanism is presented, with three revolute couplings a, b, d, belonging to class V, and one spherical coupling c, belonging to class III. Applying Reshetov method, the analysis presented Table 9 can be performed.

Table 9 – Mobility and redundant constraint analysis of mechanism in Figure 46

$F_N = 1$		
$f_{tx} = 0 \longleftarrow a$	$f_{rx} = 1$	**c**
$f_{ty} = 0 \longleftarrow b$	$f_{ry} = 1$	**c**
$F_{tz} = 0$	$f_{rz} = 4$	**abcd**
$C_N = 1$		

In Table 9 the analysis of mobility and redundant constraints of the spatial four-bar mechanism is presented. In the left-hand part of the table, linear mobilities f_{tx}, f_{ty} and f_{tz} are listed, while in the right-hand part angular mobilities f_{rx}, f_{ry} and f_{rz} are reported. The couplings which allow that specific freedom are listed to the side of each mobility. No linear mobility is directly allowed by the joints of the mechanism, thus all linear mobilities are zero in the left-hand part of the table. The three revolute couplings a, b e d allow one freedom each along z-axis, and the spherical coupling c allows three angular mobilities along each one of the coordinate axis. Thus on the right-hand side of Table 9 the number of angular mobilities are indicated, with the respective couplings.

Proposition 1 is verified as the four-bar mechanism owns all three angular mobilities. No linear mobility is found in couplings a, b, c and d but a linear mobility can be replaced by an extra angular mobility perpendicular to the direction of the linear mobility, as stated by Proposition 2. The mechanism presents four angular mobilities along z axis, thus two angular mobilities can be employed to replace linear mobilities along respectively axes x and y. In Table 9 the replacement of a missing linear mobility (on the left side) with an angular one is indicated with a zig-zag arrow leading from the angular mobility elected for replacement (on the right side) to the linear mobility. The arrows denotes the link that rotates at such replacement, and the first symbol is the kinematic pair whose angular mobility is used for the replacement. This is essential to avoid exploiting a single link twice by rotation about the same axis, although a link may be rotated about different axes. The replacement described in Table 9 for the four-bar mechanism are now described in greater details.

In Figure 47a the linear mobility along x-axis can be obtained as a rotation of link 2 around joint a. Thus in Table 9 the arrow leading from angular mobility f_{rz} to the linear one f_{tx} indicates such replacement obtained by rotation of joint a. This replacement is not unique, the same linear mobility could be replaced by the rotation of joint d for instance. The essential point is that each extra angular mobility can be employed for replacement of only a missing linear mobility.

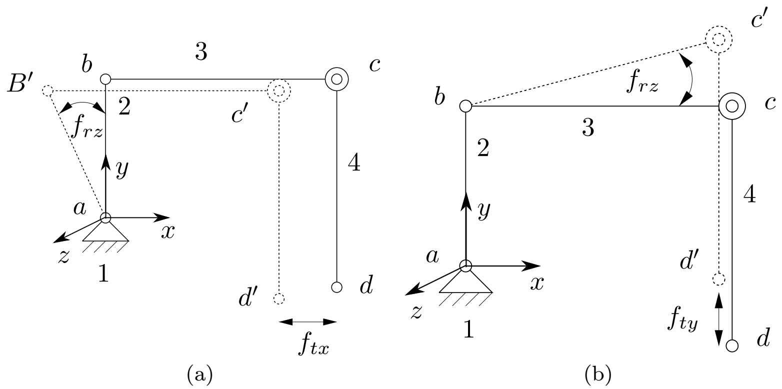


Figure 47 – Replacement of linear mobilities along x (a) and y (b) by means of rotation around z.

In the same way the missing linear mobility along y-axis can be replaced by rotation of link 3 around joint b, as showed in Figure 47b. In Table 9 this compensation is indicated by the arrow leading form f_{rz} to f_{t_y}. The missing linear mobility along z-axis cannot be compensated, because no extra angular mobility about an axis perpendicular to z is present in the mechanism. Thus the missing linear mobility along z-axis indicates a redundant constraint, which is indicated in Table 9 with the corresponding arrow. Assume that at least one freedom is left for each mobility.

Thus in Table 9, after the replacement of the missing linear mobilities f_{tx} and f_{ty}, the mechanism presents the following mobilities: linear mobilities $f_{tx} = 1$, $f_{ty} = 1$ and $f_{tz} = 0$ (redundant constraint), angular mobilities $f_{rx} = 1$, $f_{ry} = 1$ and $f_{rz} = 2$ (one mobility of the mechanism). With respect to angular mobilities around z-axis, two of the four original mobilities have been employed to replace the linear missing mobilities, one is retained for closing the loop, *i.e.* in order to satisfy Proposition 1, and one remains as mobility of the mechanism.

Therefore, as indicated in Table 9, the four-bar mechanism presents mobility $F_N = 1$ and degree of constraint $C_N = 1$. Applying Equation 5.8, with $\lambda = 6$ as the mechanism considered is spatial

$$C_N = F - \lambda \cdot \nu + C_N = 6 - 6 \cdot +1 = 1 \tag{5.10}$$

It is important to regard that, applying Grübler-Kutzbach equation, *i.e.* $F_N = \lambda(n - j - 1) + \sum_{i=1}^{j} f_i$, the mobility of the mechanism is incorrectly evaluated as $F_N = 0$. Thus mobility calculation cannot be performed without considering the number C_N of redundant restrictions.

The analysis of the four-bar mechanism presented in Table 9 indicates that linear mobility is missing along the z-axis. Thus if a misalignment δ_z along z-axis is present, as shown in Figure 48, a strain is imposed on the links of the mechanism when the loop is closed. As a consequence joints a, b and d are subjected to a torque along y-axis, which can reduce mechanism efficiency or cause fatigue failure.

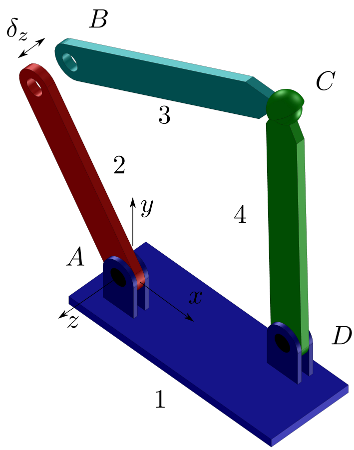


Figure 48 – Four-bar mechanism with misalignment of links

The replacement of linear mobilities presented in Table 9 is not unique. Different replacement solutions can be chosen, with the same result in terms of mobility and degree of constraint. An alternative replacement scheme is presented in Table 10.

Table 10 – Linear mobility replacement solution equivalent to Table 9

$F_N = 1$			
$f_{tx} = 0 \longleftarrow b$		$f_{rx} = 1$	**b**
$f_{ty} = 0 \longleftarrow a$		$f_{ry} = 1$	**b**
$f_{tz} = 0 \longleftarrow b$		$f_{rz} = 4$	**abcd**
$C_N = 1$			

Based on the analysis performed, the redundant restriction detected can be eliminated, by introducing additional freedoms in the joints. The revolute coupling b can be substituted by a spherical one, as shown in Figure 49.

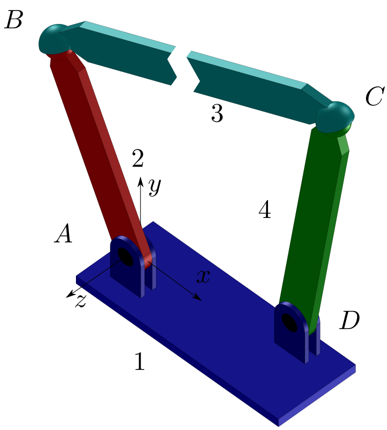


Figure 49 – Self-aligning four-bar mechanism

In Table 11 the analysis of the modified mechanism in Figure 49 is presented.

Table 11 – Analysis of mechanism presented in Figure 49

$F_N = 2$		
$f_{tx} = 0 \longleftarrow b$	$f_{rx} = 2$	**bc**
$f_{ty} = 0 \longleftarrow a$	$f_{ry} = 2$	**bc**
$F_{tz} = 0 \longleftarrow b$	$f_{rz} = 4$	**abcd**
$C_N = 0$		

When the revolute pair b is replaced by a spherical one, two additional angular mobilities are added to the mechanism. The analysis presented in Table 11 indicates that no redundant constraint is present in the mechanism, thus the mechanism is self-aligning. Moreover, one additional mobility is present in the mechanism, *i.e.* the rotation of link 3 along its axis. This mobility is defined by Reshetov(26) as *local mobility*.

The method proposed by Reshetov does not require the calculation of the motion and action screws of the mechanism. Thus it permits a quick inspection of a mechanism in order to detect redundant constraints. A complete method for multi-loop mechanism analysis can be found in Reshetov(26) and (92). It is important to regard that when this method is applied to multi-loop mechanisms, only fundamental loops of mechanism are considered. When the number of loops in the mechanism increases, the complexity of this analysis grows exponentially. However, this Thesis introduces a counterexample for the Reshetov's method, as presented in the next section.

5.3 CONTRIBUTIONS TO THE RESHETOV METHOD

In this section a proof for the Propositions 1 and 2 stated by Reshetov is first presented. Then a counterexample for the method of Reshetov is introduced. This flaw is analysed in terms of freedoms and constraints.

5.3.1 Proof of Propositions 1 and 2 by means of screw theory

Recalling that the existence of redundant constraints in a mechanism is related to the existence of circuit actions, as stated in Section 3.1, the reciprocity condition for circuit actions is herein remembered:

$$\left[\boldsymbol{\hat{M}_D^T}\right]_{F,6} \begin{bmatrix} R \\ S \\ T \\ U \\ V \\ W \end{bmatrix}_{6,1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{F,1} \tag{5.11}$$

where a circuit action can be generally represented by a wrench in axis-coordinates:

$$\boldsymbol{\$^A} = \begin{bmatrix} R \\ S \\ T \\ U \\ V \\ W \end{bmatrix}_{6,1} \tag{5.12}$$

and the order of the screw system to which all screws under consideration belong is $\lambda = 6$. Equation (5.12) states that, for a single-loop mechanism, a circuit action must be reciprocal to the motion screws of all joints of the loop. Recalling that the columns of matrix $\boldsymbol{\hat{M}_D}$ are the motion screws of the couplings, Equation (5.12) can be written as:

$$\begin{bmatrix} (\boldsymbol{\$_a^m})^T \\ (\boldsymbol{\$_b^m})^T \\ \vdots \\ (\boldsymbol{\$_e^m})^T \end{bmatrix}_{F,\lambda} \cdot \begin{bmatrix} R \\ S \\ T \\ U \\ V \\ W \end{bmatrix}_{\lambda,1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{F,1} \tag{5.13}$$

where $\$_a^m, \$_b^m, \ldots, \$_F^m$ are the motion screws associated with the joints of the mechanism. When the motion screws are written in terms of their components, Equation (5.13) becomes:

$$\begin{bmatrix} r_a & s_a & t_a & u_a & v_a & w_a \\ r_b & s_b & t_b & u_b & v_b & w_b \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ r_F & s_F & t_F & u_F & v_F & w_F \end{bmatrix}_{F,\lambda} \cdot \begin{bmatrix} R \\ S \\ T \\ U \\ V \\ W \end{bmatrix}_{\lambda,1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{F,1} \tag{5.14}$$

Equation (5.14) represents a linear homogeneous system. If the homogeneous system admits only the trivial solution the circuit action components are zero and no redundant constraint is present in the mechanism. A sufficient condition for the homogeneous system to have trivial solution is that the matrix $\hat{\boldsymbol{M}}_{\boldsymbol{D}}$ has full rank. The columns of Matrix $\hat{\boldsymbol{M}}_{\boldsymbol{D}}$ are the unit motion screws associated with the joints, in the form:

$$\hat{\$}^{\boldsymbol{m}} = \begin{bmatrix} \boldsymbol{S} \\ \boldsymbol{S_O} \times \boldsymbol{S} + h\boldsymbol{S} \end{bmatrix} = \begin{bmatrix} r \\ s \\ t \\ u \\ v \\ z \end{bmatrix} \tag{5.15}$$

where $\boldsymbol{S}$ indicated the unit direction vector or the unit angular velocity of the screw axis, $\boldsymbol{s_0}$ is the position vector of any point on the screw axis with respect to the fixed frame.

Recalling now the Proposition 1 stated by Reshetov, when one of the angular mobilities f_{rx}, f_{ry} or f_{rz} is missing, then the corresponding motion screw component r, s or t is null for all joints belonging to the loop. Thus matrix $\hat{\boldsymbol{M}}_{\boldsymbol{D}}^T$ presents a column of zeros corresponding to the missing angular mobility. Applying the *rref form* the non-pivot column, *i.e.* the column of zeros, corresponds to a free-variable, thus a redundant constraint is present in the mechanism.

In the same way, Proposition 2 can be verified. When a linear mobility f_{tx}, f_{ty} or f_{tz} is missing, the corresponding screw motion component is not necessarily null. In fact, a unit screw for a revolute joint, where the pitch is $h = 0$, can be written as:

$$\hat{\$}^{\boldsymbol{m}} = \begin{bmatrix} \boldsymbol{S} \\ \boldsymbol{S_O} \times \boldsymbol{S} \end{bmatrix} \tag{5.16}$$

Thus a linear mobility can be obtained as the rotation of an angular mobility along an axis perpendicular to the axis of the missing linear mobility. Proposition 2 is thus verified.

An example is now considered. The mechanism presented in Figure 50 is a four-bar mechanism, with $n = 4$ links and $g = 4$ joints, where a is a planar joint which allows linear mobilities along x and y-axes and angular mobility around the same axis. Joint b is a revolute coupling, joint c is a cylindrical coupling along z-axis and d is an universal coupling along y and z-axes.

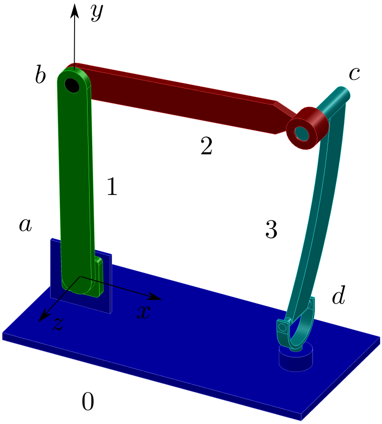


Figure 50 – Four-bar mechanism with $F_N = 3$ and $C_N = 1$.

For the mechanism in Figure 50 matrix $\boldsymbol{\hat{M}_D}$ can be written as:

$$\boldsymbol{\hat{M}_D} = \begin{array}{c} \begin{array}{cccccccc} a_t & a_u & a_v & b_t & c_t & c_w & d_s & d_t \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}_{6,8} \end{array} \tag{5.17}$$

Thus Equation (5.18) is:

$$\begin{bmatrix} 0 & \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{8,6} \begin{bmatrix} R \\ \boxed{S} \\ \boxed{T} \\ \boxed{U} \\ \boxed{V} \\ \boxed{W} \end{bmatrix}_{6,1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{8,1} \tag{5.18}$$

where matrix $\boldsymbol{\hat{M}_D^T}$ has been brought to *rref* form. It can be verified that R, which constraint the rotation around x-axis, is a free variable.

Thus the mechanism presents a redundant constraint, *i.e.* the missing mobility around x-axis.

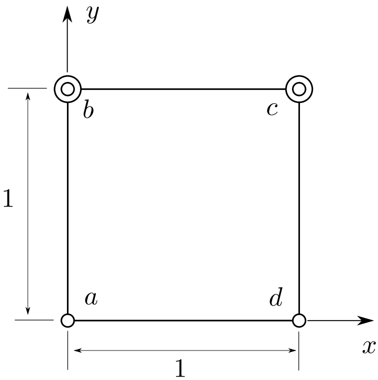


Figure 51 – Self-aligning four-bar mechanism

In Figure 51 the structural representation of the four-bar mechanism of Figure 49 is presented. The mechanism has two revolute couplings a, d and two spherical couplings b and c. The corresponding matrix $\hat{\boldsymbol{M}}_{\boldsymbol{D}}^T$ can be written as:

$$[\hat{\boldsymbol{M}}_{\boldsymbol{D}}]^T_{(8,6)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix} \tag{5.19}$$

which has full rank, thus the mechanism has no redundant constraints, as stated by the method of Reshetov.

5.3.2 Counterexample for Reshetov method and its analysis

The Reshetov method can, in principle, be applied to *Tripteron* mechanism, already analysed in Section 3.2.1.2. The Reshetov method is applied as presented in Table 12.

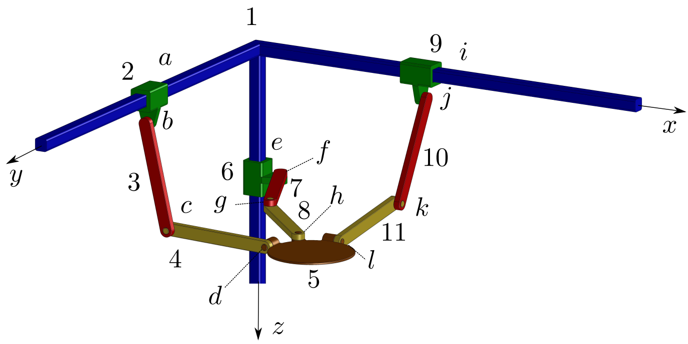


Figure 52 – *Tripteron* mechanism

Table 12 – *Tripteron* mechanism analysis

		$F_N = 2$		
		$\dot{a}\dot{b}\dot{c}\dot{d}hgfe$		
	$f_{tx} = 0 \leftarrow$	b	$f_{rx} = 0$	
a	$f_{ty} = 1$		$f_{ry} = 3$	**bcd**
e	$f_{tz} = 1$		$f_{rz} = 1$	**f**
		$efgh\dot{l}\dot{k}\dot{j}\dot{i}$		
i	$f_{tx} = 1$		$f_{rx} = 3$	**jkl**
	$f_{ty} = 0 \leftarrow$	gh	$f_{ry} = 0$	
	$f_{tz} = 0 \leftarrow$	jk	$f_{rz} = 2$	**gh**
		$C_N = 2$		

The result calculated however is not correct, as the *Tripteron* mechanism has $F_N = 3$ degree of freedom and $C_N = 3$ degree of constraint, while the analysis presented in Table 12 detects only two mobility and two redundant constraints.

The counterexample can be analysed by means of screw theory and Davies method. For a multi-loop mechanism, Equation (3.27) states that a circuit action cannot expend power on any motions allowed by the coupling of the mechanism. Alternatively, in Davies(37), the same is stated as:

If an edge belongs to only one independent circuit the characteristic screw of the joint is reciprocal to the circuit action. If an edge belongs *to more than one independent circuit the characteristic screw of the joint is reciprocal to the resultant of the circuit action of the circuits to which the edges belong.*

On the other hand Reshetov method applied for a multi-loop mechanism states each joint, which belongs to more the one independent circuit, must be reciprocal separately to the circuit actions of each circuit to which the joint belongs. Considering the *Tripteron* mechanisms, only the two fundamental loops $a-b-c-d-h-g-f-e$ and $e-f-g-h-l-k-j-i$ are considered, as stated by Reshetov method.

Applying Reshetov method it can be verified that a couple around x-axis can be locked in the first circuit, *i.e.* a circuit action whose action screw $\left[\begin{array}{cccccc} R_1 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$ is reciprocal to all motions screws of the joints of that circuit. Considering the second loop, a couple around y-axis can be locked in the circuit, *i.e.* a circuit action whose action screw
$\left[\begin{array}{cccccc} 0 & S_2 & 0 & 0 & 0 & 0 \end{array}\right]$ is reciprocal to all motions screws of the joints of that circuit.

Following the Reshetov method, a couple around z-axis, *i.e.* a circuit action whose action screw is $\left[\begin{array}{cccccc} 0 & 0 & T & U & V & W \end{array}\right]$, cannot be locked into loop , $a-b-c-d-h-g-f-e$ or $e-f-g-h-l-k-j-i$ because it is not reciprocal to the motion screws of joints i, l, m, n, which are revolute couplings around z-axis. This result in not correct, because it can be verified that two couples around z-axis, owing the same magnitude but opposite direction, can be locked at the same time in the two loops. In other words, it can be verified that two circuit actions, whose action screws are $\boldsymbol{\$}^{\boldsymbol{a}}_{\boldsymbol{T_1}} = \left[\begin{array}{cccccc} 0 & 0 & T_1 & 0 & 0 & 0 \end{array}\right]$ and $\boldsymbol{\$}^{\boldsymbol{a}}_{\boldsymbol{T_2}} = \left[\begin{array}{cccccc} 0 & 0 & T_2 & 0 & 0 & 0 \end{array}\right]$ with $T_2 = -T_1$, are reciprocal to the joint of the mechanism. More precisely, joints i, l, m, n are reciprocal to the resultant of circuit actions $\boldsymbol{\$}^{\boldsymbol{a}}_{\boldsymbol{T_1}} + \boldsymbol{\$}^{\boldsymbol{a}}_{\boldsymbol{T_2}} = 0$, where the resultant of the circuit actions is null. The same result has been found in Section 3.2.1.2.

6 CONCLUSIONS

Structure analysis of mechanism constitutes a fundamental theoretical basis for analysis in kinematics, dynamics and synthesis of mechanism. The increasing interest for low mobility overconstrained parallel mechanism calls for further theoretical research into the structure type, and kinematic characteristics of mechanisms. Moreover, the analysis of complex spatial overconstrained mechanisms requires a complete model of a mechanism in terms of screw theory, as the Grübler-Kutzbach formulation cannot correctly evaluate the mobility of these mechanisms.

Screw theory formulation has been widely employed as a means of representing mechanisms in terms of linear and angular velocities of its links, forces and couples applied by its joints. Transformation between screw-based method and matrix-based method is straightforward, thus linear algebra can be conveniently applied to screw theory formulation.

Davies' adaptation of Kirchhoff's laws to mechanical networks provides a very general formulation of freedoms and constraints in mechanisms. This formulation is based on screw theory representation for freedoms and constraints. This formulation has been adopted along this Thesis as a basis for modelling mechanisms, and further advances have been proposed herein which greatly extend the analysis of mechanisms.

In Chapter 3 a new approach for analysis of overconstrained mechanism based on screw theory is presented. The formulation for freedoms and constraints is based on the Davies' method. This approach permits a deeper understanding of mechanism's overconstraint in terms of circuit actions. A method for redundant constraint elimination is proposed, focused on multibody simulation systems.

When the complexity of a mechanism increases in terms of number of links, joints and loops, the combinatorial explosion of the set of solutions to be analysed increases exponentially. Regarding the problem of selecting a valid actuation scheme for a given mechanism, multiple valid sets of actuation can be found. On the other hand, multiple self-aligning kinematically equivalent mechanisms can be derived from an overconstrained one. This fact suggests the use of combinatorial analysis in order to investigate these aspects of mechanisms.

In Chapter 4 matroid theory is introduced as a powerful set of tools for combintorial analysis. Based on freedoms and constraints formulation for mechanism introduced in the previous chapter, important contributions are proposed for mechanism analysis by means of matroid theory.

First, a new approach for investigating actuation schemes in mechanism is presented. Based on matroid theory, a new algorithm is proposed for enumerating all valid actuation schemes for a given mechanism. The algorithm has been applied to a *3-PPRR* mechanism already analysed in literature (18) and the result compared and verified. Thus the new algorithm proposed herein correctly enumerate all feasible actuation schemes in polynomial time. Moreover, a new algorithm is introduced for selecting an optimal actuation scheme based on a set of criteria describing mechanism specifications.

On the other hand, a new algorithm for enumerating self-aligning kinematically equivalent mechanisms is introduced. This algorithm enumerates all self-aligning mechanisms derived from an overconstrained one in polynomial time. With the increase of the complexity of mechanisms, the number of self-aligning solutions grows exponentially. Thus a further new algorithm is proposed in this Thesis for selecting an optimal self-aligning kinematically equivalent mechanism, based on a set of criteria describing mechanism specifications.

The new algorithms applies matroid theory for investigating mechanisms, based on the freedom and constraint formulation introduced in this Thesis. The algorithms have all been implemented in *SAGE* software (86) and examples of application are presented along this work.

Partial results from this Thesis have been presented in the X Congreso Argentino de Mecánica Computacional- MECOM Salta (32) and in the 14^{th} IfToMM World Congress - Taipei (95).

6.1 SUGGESTIONS FOR FUTURE WORK

The following suggestions are presented for future work:

- Friction shall be included in the model of redundant constraints. When friction is considered, some self-aligning characteristics of a mechanism do not hold for every configuration (26). Laus(71) presented an extension of Davies' formulation with friction models, which can be used for redundant constraint analysis.

- Introduction of flexibility in the model of some elements of mechanism shall be addressed, in order to obtain self-aligning characteristics with parallel constraints. Wojtyra and Frączek(34) modelling of flexible bodies can be integrated inside Davies formulation.

- The new Algorithm (2) proposed in this Thesis for finding an optimal actuation scheme of a mechanism is based upon a set of criteria. An investigation on the best criteria for actuation based on mechanism specification shall be addressed.

- The new Algorithm (5) proposed herein for finding an optimal self-aligning kinematically equivalent mechanism is based upon a set of criteria. An investigation on best criteria for deriving self-aligning mechanism based on mechanism specification shall be addressed.

- For complex mechanism, multiple solutions of valid actuation scheme exist. Thus ranking all valid actuation schemes enumerated by the new algorithm proposed in this Thesis shall be addressed.

- Ranking all self-aligning kinematically equivalent enumerated by the new algorithm proposed in this Thesis, shall be addressed.

- Further combinatorial analysis of mechanisms main parameters, as connectivity and redundancy, shall be addressed applying matroid theory.

- A method for synthesis of mechanism by means of matroid theory shall be addressed.

- A mechanism design environment, based on Davies's method, with friction modelling (71) and actuation and self-aligning analysis shall be addressed.

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APPENDIX A – Linear Algebra review

In this appendix a brief review of the main topics of Linear Algebra applied in this Thesis is presented. For those interested in further reading, (68), (96), (97) and (78), (98) provide a complete theory of the mathematical tools presented, including proof of theorems herein introduced.

A.1 LINEAR SYSTEM OF EQUATIONS AND MATRICES

Consider a set of m linear equations in n real variables $x_1, \ldots, x_n$, also called a system of linear equations.

A system of linear equations can be written in the form:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ldots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m \end{array} \tag{A.1}$$

Consider the system in (A.1). The information given in the left-hand side of this system can be neatly written in terms on $m \times n$ coefficient matrix

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \tag{A.2}$$

The right-hand side of (A.1) is given by the column vector:

$$\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \tag{A.3}$$

and the n variables can be grouped in a column vector denoted by $\boldsymbol{x}$:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{A.4}$$

The linear system of (A.1) can thus be written as:

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{A.5}$$

Hence the matrix that describes completely the system (A.1) is called the augmented matrix and denoted by $[\boldsymbol{A}|\boldsymbol{b}]$.

$$[\boldsymbol{A}|\boldsymbol{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right] \tag{A.6}$$

A.1.0.1 Elementary Operations

Definition 4. *The following three operations on a given system of linear equations are called elementary:*

1. Change the order of the equations.

2. Multiply an equation by a nonzero number.

3. Add (subtract) from one equation a multiple of another equation.

Lemma 1. *Suppose that a system of linear equations, named system II, was obtained from a system of linear equations, named system I, by using a sequence of elementary operations. Then a sequence of elementary operations can be performed on system II to obtain system I. In particular, the systems I and II are equivalent,* i.e. *the two systems have the same set of solutions.*

The elementary operations on a system of linear equations described in Definition 4 are equivalent to the *Elementary row operations* on the augmented matrix corresponding to the system, as described in the next section.

A.1.1 Gauss' method and Elementary row operations

Theorem 2. *(Gauss' method) If a linear system is changed to another by one of these operations:*

1. An equation is swapped with another;

2. An equation has both sides multiplied by a nonzero constant;

3. *An equation is replaced by the sum of itself and a multiple of another.*

then the two systems have the same set of solutions.

Definition 5. *The three operations from Theorem 2 are the elementary reduction operations, or elementary row operations, or Gaussian operations. They are swapping, multiplying by a scalar (or rescaling), and row combination. Let $\boldsymbol{C}$ be given $m \times n$ matrix. Then the following three operations are called ERO (Elementary Row Operations) and denoted as follows.*

1. *Interchange the rows i and j, where $i \neq j$*

$\rho_i \longleftrightarrow \rho_j$

2. *Multiply the row i by a nonzero number $a \neq 0$*

$a \times \rho_i \longrightarrow \rho_i$

3. *Replace the row i by its sum with a multiple of the row $j \neq i$*

$\rho_i + a \times \rho_j \longrightarrow \rho_i$

Lemma 2. *The elementary row operations are reversible. More precisely:*

1. $\rho_j \longleftrightarrow \rho_i$ *is the inverse of* $\rho_i \longleftrightarrow \rho_j$
2. $\frac{1}{a} \times \rho_i \longrightarrow \rho_i$ *is the inverse of* $a \times \rho_i \longrightarrow \rho_i$
3. $\rho_i - a \times \rho_j \longrightarrow \rho_i$ *is the inverse of* $\rho_i + a \times \rho_j \longrightarrow \rho_i$

A.1.1.1 Example

Gauss' method consist of systematically applying those row operations to solve a system. A simple example is presented in the following.

$$\begin{aligned} x + y \quad\quad &= 0 \\ 2x - y \; + 3z &= 3 \\ x - 2y \; - z &= 7 \end{aligned} \tag{A.7}$$

By subtracting the first row multiplied by two from the second row, the term $2x$ of the second row is eliminated. In the same way subtracting the first row from the third one the term x is eliminated.

$$
\begin{aligned}
& & x+y \quad\quad &= 0 \\
\boldsymbol{-2\rho_1+\rho_2} &\longrightarrow & -3y+3z &= 3 \\
\boldsymbol{-\rho_1+\rho_3} &\longrightarrow & -3y\ -z &= 7
\end{aligned}
\tag{A.8}
$$

Applying again the row operations indicated

$$
\begin{aligned}
& & x+y \quad\quad &= 0 \\
& & -3y+3z &= 3 \\
\boldsymbol{-\rho_2+\rho_3} &\longrightarrow & -4z &= 4
\end{aligned}
\tag{A.9}
$$

the system can be solved:

$$
\begin{aligned}
x &= 2 \\
y &= -2 \\
z &= -1
\end{aligned}
\tag{A.10}
$$

A.1.2 Row Echelon form (ref)

Definition 6. *A matrix $\boldsymbol{C}$ is in a row echelon form (ref) if it satisfies the following conditions:*

1. *The first nonzero entry in each row is 1. This entry is called a pivot.*

2. *If row k does not consists entirely of zeros, then the number of leading zero entries in row $k+1$ is greater than the number of leading zeros in row k.*

3. *Zero rows appear below the rows having nonzero entries.*

Lemma 3. *Every matrix can be brought to a ref using ERO.*

A.1.2.1 Example

Consider the following matrix:

$$
\begin{bmatrix} 1 & 3 & -1 \\ 2 & 7 & 4 \\ 0 & 2 & -2 \end{bmatrix}
\xrightarrow{\text{add -2}\times\rho_1 \text{ to } \rho_2}
\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & -2 \end{bmatrix}
\tag{A.11}
$$

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & -2 \end{bmatrix} \xrightarrow{\text{add -2 } \times\rho_2 \text{ to } \rho_3} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -6 \end{bmatrix} \xrightarrow{\text{divide by } -6 \ \rho_3} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

by means of the elementary operation the matrix is expressed in row echelon form as Definition 6. Note that the *ref* of a matrix is not unique in general. For example by using elementary row operation of the form $\rho_1 - a \times \rho_2 \longrightarrow \rho_1$ for $a \neq 0$ it is always possible to bring the above matrix in the row echelon form to another matrix in a row echelon form.

Definition 7. *Let $\boldsymbol{C} = [a_{ij}]$ be an $m \times n$ matrix in a ref. Then the number of pivots is the rank r of matrix $\boldsymbol{C}$.*

Lemma 4. *Let $\boldsymbol{C} = [c_{ij}]$ be an $m \times n$ matrix in a ref with rank r.*

1. *Rank of matrix $\boldsymbol{C}$ is $r = 0$ if and only if $\boldsymbol{C} = 0$.*

2. *Rank of matrix $\boldsymbol{C}$ is $r \leq min(m, n)$.*

3. *If $m > r$ then the last $m - r$ rows of matrix $\boldsymbol{C}$ are zero rows.*

Lemma 5. *Let $\boldsymbol{B}$ be an $m \times n$ matrix and assume that $\boldsymbol{B}$ can be brought to a row echelon matrix $\boldsymbol{C}$. Then rank r and the location of the pivots do not depend on a particular choice of the (ref) matrix $\boldsymbol{C}$.*

A.1.3 Reduced Row Echelon Form (rref)

Among all row echelon forms $\boldsymbol{C}$ of a given matrix $\boldsymbol{B}$ there is one special *ref* which is called reduced row echelon form and denoted by *rref*.

Definition 8. *Let $\boldsymbol{C}$ be a matrix in a row echelon form. Then $\boldsymbol{C}$ is an rref if 1 is a pivot on the column k of $\boldsymbol{C}$ then all other elements on the column k of $\boldsymbol{C}$ are zero.*

A.1.3.1 Example of matrix in *rref*

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 1 & 7 \\ 0 & 1 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.12}$$

Theorem 3. *Let A be an $m \times n$ matrix. Then it rref is unique.*

The process of bringing a matrix to *rref* is called Gauss-Jordan reduction. The following algorithm, called Gauss-Jordan algorithm, can be applied.

A.1.3.2 Gauss-Jordan algorithm for *rref*

Algorithm 6 Gauss-Jordan algorithm

1: input: matrix $\boldsymbol{C}$ $m \times n$
2: $i \leftarrow 1$
3: $j \leftarrow 1$
4: **if** $c_{ij} = 0$ **then**
5:     swap row i with a row k with $k > i$ and $a_{ki} \neq 0$
6:     **if** all entries in the column are zero **then**
7:         $j \leftarrow j + 1$
8: Divide row i by c_{ij} to make the pivot entry $= 1$
9: Eliminate all other entries in the column j by subtracting suitable multiples from the other rows.
10: $i \leftarrow i + 1$
11: $j \leftarrow j + 1$
12: **go to** 4

The algorithm stops after we process the last row or the last column of the matrix. The output of the Gauss-Jordan algorithm is the matrix in reduced row-echelon form.

A.1.4 Solution of linear homogeneous system

Definition 9. *The linear system system $\boldsymbol{Ax} = \boldsymbol{b}$ is called homogeneous if $\boldsymbol{b} = \boldsymbol{0}$,* i.e. $b_1 = \ldots = b_m = 0$. *A homogeneous system of linear equations has always a solution $\boldsymbol{x} = \boldsymbol{0}$, which is called a trivial solution.*

Definition 10. *Consider an $m \times n$ matrix $\boldsymbol{A}$, the null space of $\boldsymbol{A}$ is nullspace($\boldsymbol{A}$) $= \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\}$. Its dimension is referred to as the nullity of $\boldsymbol{A}$.*

Theorem 4. *Fundamental theorem of linear homogeneous system . Given a homogeneous linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ with m equations and n unknowns:*

1. If rank($\boldsymbol{A}$) $= n$, then the system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ has only the trivial solution $\boldsymbol{x} = 0$, so nullspace($\boldsymbol{A}$) $= \emptyset$.

2. If rank($\boldsymbol{A}$) $= r < n$, then $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ has an infinite number of solutions and the general solution will contain $n - r$ arbitrary constants. All the solutions are of the form:

$$\boldsymbol{x} = c_1\boldsymbol{x_1} + c_2\boldsymbol{x_2} + \ldots + c_{n-r}\boldsymbol{x_{n-r}}, \tag{A.13}$$

where $\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_{n-r}}$ is a basis for nullspace($\boldsymbol{A}$).

As discussed in the previous section a linear homogeneous system (its respective matrix) can always be brought in the following form:

$$\left[\begin{array}{ccccccccccccc}
p_1 & * & \cdots & & & & & & \cdots & & & & * \\
0 & \cdots & & p_2 & * & \cdots & & & & & & & \\
 & & & 0 & \cdots & & p_3 & * & & & & & \\
\vdots & & & & & & 0 & \cdots & & & & & \vdots \\
 & & & & & & & & \ddots & & & & \\
 & & & & & & & & & p_r & * & \cdots & * \\
0 & \cdots & & & & & & & & 0 & \cdots & & 0 \\
\vdots & & & & & & & & & \vdots & & & \vdots \\
0 & \cdots & & & & & & & & 0 & \cdots & & 0
\end{array}\right]
\begin{array}{l}
\left.\begin{array}{l} \\ \\ \\ \\ \\ \\ \end{array}\right\} r\text{: rank} \\
\left.\begin{array}{l} \\ \\ \\ \end{array}\right\} m-r
\end{array} \tag{A.14}$$

where the nonzero coefficients $p_1, p_2, \ldots, p_r$ form a staircase pattern. These coefficients are called the *pivot values* and are placed in the *pivot positions.* By definition , the rank is the number of nonzero lines, which are listed first. If $r < m$, the last $m - r$ lines are filled with zeros. The increasing integers $1 < j_1 < j_2 < \ldots < j_r$ are the indices of the pivots columns, which correspond to the *pivot variables* $x_{j_1}, x_{j_2}, \ldots, x_{j_r}$. As already mentioned, the rank r is always $r \leq min(m, n)$. If $r < n$, there are $n - r$ *nonpivot variable*s, called *free variables.*

Once given arbitrary values to the free variables, the value of the last pivot variable x_r can be deduced. Working upwards, all values of the pivots values can be than determined. This method is called *back-substitution* procedure and leads to a general solution of the system.

For a better understanding an example is introduced. Consider the following problem: compute the nullspace of matrix $\boldsymbol{A}$, *i.e.* find the dimension and a basis.

$$[\boldsymbol{A}] = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}_{4,5} \end{array} \tag{A.15}$$

In order to solve the problem, it suffices to remember that the nullspace of $\boldsymbol{A}$ is the set of vector $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}^T$ solution of $\boldsymbol{Ax} = \boldsymbol{0}$, as by Definition 10. Matrix $\boldsymbol{A}$ is brought to *rref* by use of elementary row operations, in the form:

$$[\boldsymbol{A}] = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} \boxed{1} & 0 & 0 & -2 & 3 \\ 0 & \boxed{1} & 0 & -1 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{4,5} \end{array} \tag{A.16}$$

where the boxed 1 are *pivots*, whose position denotes the *pivot columns*. Therefore columns 1, 2 and 3 are pivot columns, which correspond to the pivot variables x_1, x_2 and x_3, or leading variables. The *non-pivot variables* x_4 and x_5 are *free-variables* and may be chosen arbitrarily. The general solution of the system $\boldsymbol{Ax} = \boldsymbol{0}$ can be written in terms of x_4 and x_5 by back substitution:

$$\begin{aligned} x_1 &= 2r - 3s \\ x_2 &= r \\ x_3 &= -r \\ x_4 &= r \\ x_5 &= s \end{aligned} \tag{A.17}$$

The nullspace of $\boldsymbol{A}$ has dimension 2 and a basis can be written attributing the value 1 to the first free-variable and 0 to the second:

$$\{\{2, 1, -1, 1, 0\}, \{-3, 0, 0, 0, 1\}\} \tag{A.18}$$

As stated in Theorem (4), a linear homogeneous system having more variables than equations admits a nontrivial solution. On the other hand, if $rank(\boldsymbol{A}) = n$ there are no free variables. Consider the following homogeneous system:

$$
\left\{\begin{array}{ccccccc}
x_1 & + & 3x_2 & - & 5x_3 & = & 0 \\
x_1 & - & 2x_2 & + & 7x_3 & = & 0 \\
2x_1 & + & x_2 & - & x_3 & = & 0
\end{array}\right. \tag{A.19}
$$

with coefficient matrix $\boldsymbol{A}$:

$$
\begin{array}{c}
\begin{array}{ccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 \end{array} \\
\begin{bmatrix}
1 & 3 & -5 \\
1 & -2 & 7 \\
2 & 1 & -1
\end{bmatrix}_{3,3}
\end{array} \tag{A.20}
$$

Applying *rref* to matrix $\boldsymbol{A}$

$$
\begin{array}{c}
\begin{array}{ccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 \end{array} \\
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}_{3,3}
\end{array} \tag{A.21}
$$

a pivot is found in each column, as expected, because rank($\boldsymbol{A}$) is equal to the number of pivots. Thus the system has only the trivial solution in this case.

In general the coefficient matrix $\boldsymbol{A}$ of a linear homogeneous system can therefore be transformed in a row-reduced array in the form:

$$
\left[\begin{array}{cccccccc}
\boxed{p_1} & * & * & * & * & * & * & * \\
0 & \boxed{p_2} & * & * & * & * & * & * \\
\vdots & & & \boxed{p_3} & * & * & * & * \\
 & & & & \ddots & * & * & * \\
 & & & & & \boxed{p_r} & * & * \\
0 & & & & & & & 0
\end{array}\right]
\begin{array}{l}
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\} r\text{: rank} \\
\left.\vphantom{1}\right\} m-r
\end{array} \tag{A.22}
$$

where the squares contain the pivots values with $p_i \neq 0$ $(1 \leq i \leq r)$, while the $*$ may contain any entry. Multiplying the first row by $1/p_1$, the second by $1/p_2$ and so on, the matrix is reduced in row-echelon form:

$$\begin{bmatrix} \boxed{1} & * & * & * & * & * & * & * \\ 0 & \boxed{1} & * & * & * & * & * & * \\ \vdots & & & \boxed{1} & * & * & * & * \\ & & & & \ddots & * & * & * \\ & & & & & \boxed{1} & * & * \\ 0 & & & & & & & 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{matrix}}\right\} r\text{: rank} \\ \left.\vphantom{1}\right\} m-r \end{matrix} \tag{A.23}$$

where the pivot values are 1's, as in Definition (6). By subtracting a suitable multiple of the second row from the first one, suitable multiples of the third from the first and second rows, and so on, the reduced-row echelon form can be obtained:

$$\begin{bmatrix} \boxed{1} & 0 & * & 0 & * & 0 & * & * \\ 0 & \boxed{1} & * & 0 & * & 0 & * & * \\ \vdots & & & \boxed{1} & * & 0 & * & * \\ & & & & \ddots & 0 & * & * \\ & & & & & \boxed{1} & * & * \\ 0 & & & & & & & 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{matrix}}\right\} r\text{: rank} \\ \left.\vphantom{1}\right\} m-r \end{matrix} \tag{A.24}$$

where all coefficients above a pivot postion are 0's, as in Definition (8).

A.1.4.1 Row equivalence of matrices

Definition 11. *Let $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m \times n}$. $\boldsymbol{B}$ is called row equivalent to $\boldsymbol{A}$, denoted by $\boldsymbol{B} \sim \boldsymbol{A}$, if $\boldsymbol{B}$ can be obtained from $\boldsymbol{A}$ using ERO.*

Theorem 5. *Let $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m \times n}$. Then:*

1. $\boldsymbol{B} \sim \boldsymbol{A} \iff \boldsymbol{A} \sim \boldsymbol{B}$.
2. *$\boldsymbol{B} \sim \boldsymbol{A}$ if and only if $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same rref*

A.1.5 Vector Spaces

Definition 12. *A set $\boldsymbol{V}$ is called a vector space if:*

1. *For each $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V}$, $\boldsymbol{x} + \boldsymbol{y}$ is an element of $\boldsymbol{V}$. (Addition)*

2. *For each* $\boldsymbol{x} \in \boldsymbol{V}$ *and* $a \in \mathbb{R}$, $a\boldsymbol{x}$ *is an element of* $\boldsymbol{V}$. *(Multiplication by scalar)*

The two operations satisfy the following laws:

1. $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$, *commutative law;*
2. $(\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z} = \boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z})$, *associative law;*
3. $\boldsymbol{x} + \mathbf{0} = \boldsymbol{x}$ *for each* $\boldsymbol{x}$, *neutral element* $\mathbf{0}$*;*
4. $\boldsymbol{x} + (-\boldsymbol{x}) = \mathbf{0}$, *unique anti element;*
5. $a(\boldsymbol{x} + \boldsymbol{y}) = a\boldsymbol{x} + a\boldsymbol{y}$ *for each* $\boldsymbol{x}$, y, *distributive law;*
6. $(a + b)\boldsymbol{x} = a\boldsymbol{x} + b\boldsymbol{x}$, *distributive law;*
7. $(ab)\boldsymbol{x} = a(b\boldsymbol{x})$, *associative law;*
8. $1\boldsymbol{x} = \boldsymbol{x}$ *unitary law.*

A.1.5.1 Examples of vector spaces

1. $\mathbb{R}$ - Real Line.
2. $\mathbb{R}^2$ - Plane.
3. $\mathbb{R}^3$ - Three dimensional space.
4. $\mathbb{R}^n$ - n-dimensional space.
5. $\mathbb{R}^{m \times n}$ - The space of $m \times n$ matrices.

A.1.5.2 Subspaces

Definition 13. *Let* $\boldsymbol{V}$ *be a vector space. A subset* $\boldsymbol{W}$ *of* $\boldsymbol{V}$ *is called a subspace of* $\boldsymbol{V}$ *if the following two conditions hold:*

1. *For any* $\boldsymbol{x}, \boldsymbol{y} \in W \Rightarrow \boldsymbol{x} + \boldsymbol{y} \in \boldsymbol{W}$,
2. *For any* $\boldsymbol{x} \in \boldsymbol{W}, a \in \mathbb{R} \Rightarrow a\boldsymbol{x} \in \boldsymbol{W}$.

A.1.5.3 Examples of subspaces

1. For $A \in R^{m\times n}$ the null space of $\boldsymbol{A}$, denoted by $N(\boldsymbol{A})$, is a subspace of $\mathbb{R}^n$ consisting of all vectors $\boldsymbol{x} \in \mathbb{R}^n$ such that $\boldsymbol{Ax} = \boldsymbol{0}$. Note: $N(\boldsymbol{A})$ is also called the kernel of $\boldsymbol{A}$, and denoted by ker $\boldsymbol{A}$.

2. For $A \in R^{m\times n}$ the range of $\boldsymbol{A}$, denoted by $R(\boldsymbol{A})$, is a subspace of R^m consisting of all vectors $\boldsymbol{y} \in \mathbb{R}^m$ such that $\boldsymbol{y} = \boldsymbol{Ax}$ for some $\boldsymbol{x} \in \mathbb{R}^n$.

A.1.5.4 Linear combination

For $\boldsymbol{v_1}, \ldots, \boldsymbol{v_k} \in \boldsymbol{V}$ and $a_1, \ldots, a_k \in \mathbb{R}$ the vector $a_1\boldsymbol{v_1} + a_2\boldsymbol{v_2} + \ldots + a_k\boldsymbol{v_k}$ is called a linear combination of $\boldsymbol{v_1}, \ldots, \boldsymbol{v_k}$. The set of all linear combinations of $\boldsymbol{v_1, \ldots, v_k}$ is called the span of $\boldsymbol{v_1}, \ldots, \boldsymbol{v_k}$ and denoted by span$(\boldsymbol{v_1, \ldots, v_k})$.

Proposition 3. *The span$(\boldsymbol{v_1, \ldots, v_k})$ is a linear subspace of $\boldsymbol{V}$.*

A.1.5.5 Spanning sets of vector spaces

The set $(\boldsymbol{v_1, \ldots, v_k})$ is called a spanning set of $\boldsymbol{V} \iff \boldsymbol{V} = span(\boldsymbol{v_1}, \ldots, \boldsymbol{v_k})$.

Theorem 6. *$(\boldsymbol{v_1, \ldots, v_k})$ is a spanning set of $\mathbb{R}^n \iff k \geq n$ and ref of $A = [\boldsymbol{v_1 v_2 \ldots v_k}] \in \mathbb{R}^{n\times k}$ has n pivots.*

Corollary 1. *Let $\boldsymbol{v_1}, \ldots, \boldsymbol{v_n} \in \mathbb{R}^m$. Form $\boldsymbol{A} = [\boldsymbol{v_1 v_2} \ldots \boldsymbol{v_n}] \in \mathbb{R}^{m\times n}$. Let $\boldsymbol{B} \in \mathbb{R}^{m\times n}$ be ref of $\boldsymbol{A}$. Then span $\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}$ is spanned by $\boldsymbol{v_{j1}, \ldots, v_{jr}}$ corresponding to the columns of $\boldsymbol{B}$ at which the pivots are located.*

Corollary 2. *Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and assume that $\boldsymbol{B} \in \mathbb{R}^{m\times n}$ be ref of $\boldsymbol{A}$. Then $R(\boldsymbol{A})$, the column space of $\boldsymbol{A}$, is spanned by the columns of $\boldsymbol{A}$ corresponding to the columns of $\boldsymbol{B}$ at which the pivots are located.*

Corollary 3. *Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and assume that $\boldsymbol{B} \in \mathbb{R}^{m\times n}$ be ref of $\boldsymbol{A}$. Then Col($\boldsymbol{A}$), the column space of $\boldsymbol{A}$, is spanned by the columns of $\boldsymbol{A}$ corresponding to the columns of $\boldsymbol{B}$ at which the pivots are located.*

A.1.5.6 Linear Independence

Definition 14. *A subset of a vector space is linearly independent if none of its elements is a linear combination of the others. Otherwise it is linearly dependent.*

Vectors $\boldsymbol{v_1}, \ldots, \boldsymbol{v_n} \in \boldsymbol{V}$ are linearly independent $\iff$ the equality $a_1\boldsymbol{v_1} + a_2\boldsymbol{v_2} + \ldots + a_n\boldsymbol{v_n} = \mathbf{0}$ implies that $a_1 = a_2 = \ldots = a_n = 0$.

Equivalently $\boldsymbol{v_1}, \ldots, \boldsymbol{v_n} \in \boldsymbol{V}$ are linearly independent $\iff$ every vector in $\mathrm{span}(\boldsymbol{v_1}, \ldots, \boldsymbol{v_n})$ can be written as a linear combination of $(\boldsymbol{v_1}, \ldots, \boldsymbol{v_n})$ in a unique way.

Vectors $\boldsymbol{v_1}, \ldots, \boldsymbol{v_n} \in \boldsymbol{V}$ are linearly dependent $\iff \boldsymbol{v_1}, \ldots, \boldsymbol{v_n} \in \boldsymbol{V}$ are not linearly independent.

Equivalently $\boldsymbol{v_1}, \ldots, \boldsymbol{v_n} \in \boldsymbol{V}$ are linearly dependent $\iff$ there exists a nontrivial linear combination of $\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}$ which equals to zero vector: $a_1\boldsymbol{v_1} + \ldots + a_n\boldsymbol{v_n} = \mathbf{0}$ and $|a_1| + \ldots + |a_n| > 0$.

Proposition 4. *Let $\boldsymbol{v_1}, \ldots, \boldsymbol{v_n} \in \mathbb{R}^m$. Form $A = [\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}] \in \mathbb{R}^{m\times n}$. Then $\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}$ are linearly independent. (I.e $\iff \boldsymbol{Ax} = \mathbf{0}$ has only the trivial solution. $\iff$ ref of $\boldsymbol{A}$ has n pivots).*

A.1.5.7 Basis and dimension

Definition 15. *$\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}$ form a basis in $\boldsymbol{V}$ if $\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}$ are linearly independent and span $\boldsymbol{V}$.*

Theorem 7. *Assume that $\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}$ spans $\boldsymbol{V}$. Then any collection of m vectors $\boldsymbol{u_1}, \ldots, \boldsymbol{u_m} \in \boldsymbol{V}$, such that $m > n$ is linearly dependent.*

Definition 16. *$\boldsymbol{V}$ is called n-dimensional, if $\boldsymbol{V}$ has a basis consisting of n -elements. The dimension of $\boldsymbol{V}$ is n, which is denoted by $\dim \boldsymbol{V}$. The dimension of the trivial space $\emptyset$ is 0.*

Theorem 8. *Let $dim\boldsymbol{V} = n$. Then:*

1. *Any set of n linearly independent vectors $\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}$ is a basis in $\boldsymbol{V}$.*

2. *Any set of n vectors $\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}$ that span $\boldsymbol{V}$ is a basis in $\boldsymbol{V}$.*

Theorem 9. *Let $dim\boldsymbol{V} = n$. Then:*

1. *No set of less than n vectors can span $\boldsymbol{V}$.*

2. *Any spanning set of more than n vectors can be paired down to form a basis for $\boldsymbol{V}$.*

3. *Any subset of less than n linearly independent vectors can be extended to basis of $\boldsymbol{V}$.*

A.1.5.8 Row and column spaces of matrices

Definition 17. *Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$.*

1. *Let $\boldsymbol{r_1}, \ldots, \boldsymbol{r_m} \in \mathbb{R}^n$ be the m rows of $\boldsymbol{A}$. Then the row space of $\boldsymbol{A}$ is span $(\boldsymbol{r_1}, \ldots, \boldsymbol{r_m})$ a subspace of $\mathbb{R}^n$.*

2. *Let $\boldsymbol{c_1}, \ldots, \boldsymbol{c_n} \in \mathbb{R}^m$ be the n columns of $\boldsymbol{A}$. Then the column space of $\boldsymbol{A}$ is span $(\boldsymbol{c_1}, \ldots, \boldsymbol{c_n})$ a subspace of $\mathbb{R}^m$.*

Proposition 5. *Let $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m \times n}$ and assume that $\boldsymbol{A} \sim \boldsymbol{B}$. Then $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same row spaces.*

Theorem 10. *Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{B}$ be its ref. Then:*

1. *A basis of the row space of A, which is a basis for $R(\boldsymbol{A}^T)$, consists of nonzero rows in $\boldsymbol{B}$. dim $R(\boldsymbol{A}^T)$ = rank $\boldsymbol{A}$ is number of lead variables.*

2. *A basis of column space of $\boldsymbol{A}$ consists of the columns of $\boldsymbol{A}$ in which the pivots of $\boldsymbol{B}$ are located. Hence dim$R(\boldsymbol{A})$ = rank $\boldsymbol{A}$.*

3. *A basis of the null space of $\boldsymbol{A}$ is obtained by letting each free variable to be equal 1 and all the other free variable equal to 0, and then finding the corresponding solution of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$. The dimension of $N(\boldsymbol{A})$, called the nullity of $\boldsymbol{A}$, is the number of free variables: nullity$(\boldsymbol{A}) :=$ dim$(N(\boldsymbol{A})) = n -$ rank$(\boldsymbol{A})$.*

Theorem 11. *For any $m \times n$ matrix $\boldsymbol{A}$:*

$$\text{rank}(\boldsymbol{A}) + \text{nullity}(\boldsymbol{A}) = n \tag{A.25}$$

APPENDIX B – Matroid Theory

A brief review of the main concepts matroid theory is herein presented, including main definitions and theorems.

For those interested in further reading (81), (90), (99), (100), (80) and (101) provide a complete theory of the mathematical tools presented, including proof of theorems herein introduced. A brief but complete introduction to matroid theory can be found in (102).

Matroids were first described in 1935 by the mathematician Whitney(79) as a combinatorial generalization of linear independence of vectors; *matroid* means *something sort of like a matrix*. Matroids are combinatorial structures that generalize the notion of linear independence in matrices. They play an essential role in combinatorial optimisation. Moreover, recognising that a problem can be modelled with matroids guarantees that certain algorithms will return an optimal solution.

This chapter is organised into two main sections: in Section B.1 an introduction to matroids theory and the essential definitions and algorithms employed in this Thesis are presented. For readers interested in learning wider aspects of matroids, Section B.2 presents some further concepts and more general aspect of matroid theory.

B.1 INTRODUCTION TO MATROID THEORY

There are many equivalent ways to define a matroid, the most significant being in terms of independent sets, bases, circuits, closed sets or flats, closure operators, and rank functions, each related to the concept of independence. Matroids have naturally arisen from shared behaviours of vector spaces and graph. Thus two examples in linear algebra and graph theory are first introduced, then formal definitions are presented.

The following finite collection of vectors of $\mathbb{R}^3$, arranged as column vectors in matrix $\boldsymbol{A}$, is considered:

$$\boldsymbol{A} = \begin{array}{c} \begin{array}{ccccccc} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \boldsymbol{v_4} & \boldsymbol{v_5} & \boldsymbol{v_6} & \boldsymbol{v_7} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}_{3,7} \end{array} \tag{B.1}$$

Vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_7}$ can be checked for linear independence over the field $\mathbb{R}^3$ and different sets of linearly independent vectors can be enumerated. For example sets $\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ and $\{\boldsymbol{v_3}, \boldsymbol{v_6}\}$ are linearly independent sets, while vector $\boldsymbol{v_7}$ can not be included in any linearly independent set.

A maximal independent set can be defined as a linearly independent set which cannot be contained within any other independent set of vectors. For the vectors considered the size of a maximal set can be no larger than three. The complete set $\mathcal{B}$ of all maximal independent sets can be written as:

$$\begin{aligned}\mathcal{B} = \{&\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}, \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_4}\}, \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_5}\}, \{\boldsymbol{v_1}, \boldsymbol{v_3}, \boldsymbol{v_5}\}, \{\boldsymbol{v_1}, \boldsymbol{v_4}, \boldsymbol{v_5}\},\\ &\{\boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\}, \{\boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_6}\}, \{\boldsymbol{v_2}, \boldsymbol{v_4}, \boldsymbol{v_5}\}, \{\boldsymbol{v_2}, \boldsymbol{v_4}, \boldsymbol{v_6}\}, \{\boldsymbol{v_2}, \boldsymbol{v_5}, \boldsymbol{v_6}\},\\ &\{\boldsymbol{v_3}, \boldsymbol{v_4}, \boldsymbol{v_5}\}, \{\boldsymbol{v_3}, \boldsymbol{v_5}, \boldsymbol{v_6}\}, \{\boldsymbol{v_4}, \boldsymbol{v_5}, \boldsymbol{v_6}\}\}\end{aligned} \tag{B.2}$$

It is important to regard that each maximal set contains exactly three elements. Two properties can be now stated for the maximal sets:

1. No maximal independent set can be properly contained in another maximal independent set.

2. Given any pair of elements, $B_1, B_2 \in \mathcal{B}$, if an element $\boldsymbol{v_i}$ of B_1 is removed, there is an element $\boldsymbol{v_j} \in B_2$ such that $((B_1 \backslash \boldsymbol{v_i}) \cup \boldsymbol{v_j}) \in \mathcal{B}$,
   i.e. $(B_1 \backslash \boldsymbol{v_i}) \cup \boldsymbol{v_j}$ is also a maximal independent set.

For example considering the maximal sets $B_1 = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ and $B_2 = \{\boldsymbol{v_1}, \boldsymbol{v_3}, \boldsymbol{v_5}\}$, if element $\boldsymbol{v_3}$ is removed from B_1, an element $\boldsymbol{v_5} \in B_2$ exists such that that $(B_1 \backslash \boldsymbol{v_3}) \cup \boldsymbol{v_5} = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_5}\} \in \mathcal{B}$.

An example from graph theory is now considered. First it is necessary to define the concept of independence in a graph. There are two common ways of defining independence in a graph, one based on vertices and one based on edges. The independence in terms of edges is herein introduced.

The connected graph G presented in Figure 53 is considered, whose set of edges is $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$.

A set of edges is considered independent if, for each edge, the removal of that edge makes some vertices, which was connected by the set of edges, disconnected. For example the subset of edges $\{e_1, e_3, e_4, e_5\}$, connecting vertices $\{1, 2, 3, 4\}$ is not independent, since edge e_4 can be removed form S without disconnecting any vertex. Moreover, the set S contains the closed path e_1, e_3, e_4. If some set of edges contains a closed path, also called a cycle, it cannot be an independent set of edges. On the other hand, $\{e_2, e_6\}$ connecting vertices $\{1, 2, 3\}$ is an independent

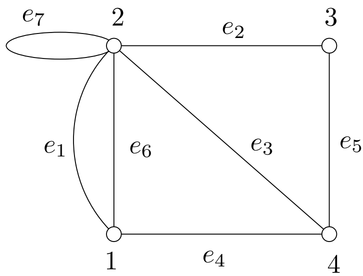


Figure 53 – Connected graph G

set. The set $\{e_7\}$ cannot be included in any independent set, because edges e_7 forms a cycle.

In a graph, a set of edges forming a tree is independent: by definition a tree does not contain a cycle and the removal of any edge from a tree disconnects some vertices. A maximal independent set is thus a set of edges which contains no cycle and also makes all vertices connected. Such set is called a spanning tree. Any connected graph has at least one spanning tree. For the connected graph G of Figure 53, the set $\mathcal{T}$ of all spanning tree is:

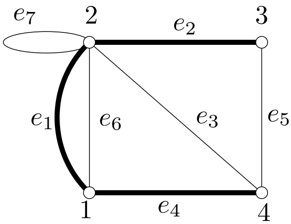
$$\begin{aligned}\mathcal{T} = \{&\{e_1,e_2,e_3\},\{e_1,e_2,e_4\},\{e_1,e_2,e_5\},\{e_1,e_3,e_5\},\{e_1,e_4,e_5\},\\ &\{e_2,e_3,e_4\},\{e_2,e_3,e_6\},\{e_2,e_4,e_5\},\{e_2,e_4,e_6\},\{e_2,e_5,e_6\},\\ &\{e_3,e_4,e_5\},\{e_3,e_5,e_6\},\{e_4,e_5,e_6\}\}\end{aligned} \tag{B.3}$$

It is important to regard that the spanning trees of a graph G have exactly the same size. Two properties can be now stated for the spanning trees:

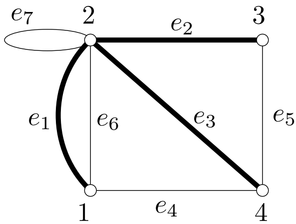
1. No spanning tree can be properly contained in another spanning tree.

2. Given any pair of spanning trees, $T_1, T_2 \in \mathcal{T}$, if an element e_i of T_1 is removed, there is an element $e_j \in T_2$ such that $((T_1 \backslash t_i) \cup t_j) \in \mathcal{T}$, *i.e.* $(T_1 \backslash t_i) \cup t_j$ is also a spanning tree.

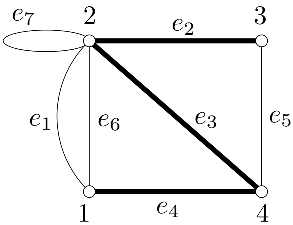
For example, the spanning trees $T_1 = \{e_1, e_2, e_4\}$ and $T_2 = \{e_1, e_2, e_3\}$ are considered, as presented in Figures 54a and 54b.



(a) Spanning tree T_1



(b) Spanning tree T_2



(c) Spanning tree T_3

Figure 54 – Spanning trees T_1, T_2 and T_3 of graph G

If edge e_1 is removed from T_1 and edge $e_3 \in T_2$ is included, than $(T_1 \backslash e_1) \cup e_3$ is also a spanning tree T_3, shown in Figure 54c.

The properties presented for maximal independent sets of a collection of vectors and spanning trees of a graph can now be generalised in the concept o matroid.

Definition 18. *A matroid $\mathcal{M}$ is an ordered pair, $(E, \mathcal{B})$, of a finite set E, called the ground set of the matroid, and a nonempty collection of bases $\mathcal{B}$ of subsets of E satisfying the following axioms:*

- *No basis properly contains another basis.*
- *If B_1 and B_2 are in $\mathcal{B}$ and $e \in B_1$, then there is an element $f \in B_2$ such that $((B1 \backslash e) \cup f) \in \mathcal{B}$.*

The bases of the matroid $\mathcal{M}$ are its maximal independent sets. Recalling the previous examples, two ways of defining matroid are presented: one based on vectors and the other based on graph.

In terms of a finite set of vectors, arranged as columns of a matrix $\boldsymbol{A}$, a matroid can be defined as:

Definition 19. *Given a matrix $\boldsymbol{A}$, a matroid $\mathcal{M}(\boldsymbol{A})$ is the matroid whose ground set E, is the set of columns of the matrix and whose set of bases, $\mathcal{B}$, is the set of maximal linearly independent sets.*

Thus for matrix $\boldsymbol{A}$ of equation (B.1), a matroid $\mathcal{M}(\boldsymbol{A})$ can be defined. The ground set of $E(\boldsymbol{A})$ is the set of columns of $\boldsymbol{A}$, *i.e.* $E(\boldsymbol{A}) = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}, \boldsymbol{v_5}, \boldsymbol{v_6}, \boldsymbol{v_7}\}$. The bases of matroid $\mathcal{M}(\boldsymbol{A})$ are the independent maximal sets, listed in Equation (B.2).

In terms of a graph a matroid can be defined as:

Definition 20. *Given a connected graph G with edge set E, the cycle matroid of G, denoted $\mathcal{M}(G)$, is the matroid whose ground set E is the set of edges of the graph and whose set of bases $\mathcal{B}$ is the set of spanning trees of G.*

Thus for the graph G of Figure 53, the ground set E is $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$. The bases of matroid $\mathcal{M}(G)$ are the spanning tree, listed in Equation (B.3).

Some further concepts and definitions for matroids are now introduced. Recalling the examples presented, it can be noticed any subset of an independent set is independent: a subset of a set of independent vectors is also independent and a subset of a set of edges which contains no cycle also contains no cycle. This property can be extended for all matroid.

Property 1. *Any set of elements of the matroid that is contained in a basis is an independent set of the matroid. Further, any independent set can be extended to a basis. Moreover, given two independent sets of different sizes, say $|I_1| < |I_2|$, then it is always possible to find some element of the larger set to include with the smaller so that it is also independent: there exists some $e \in I_2$ such that $I_1 \cup e$ is independent.*

The set of all independent set of a matroid $\mathcal{M}$ is denoted as $\mathcal{I}(\mathcal{M})$. A subset of E that is not independent is called dependent. A minimal dependent set is a circuit and can be defined as:

Definition 21. *A circuit of a matroid is a minimal dependent set. The collection of circuits of $\mathcal{M}$ is denoted $\mathcal{C}(\mathcal{M})$.*

A minimal dependent set means that any proper subset of this set is not dependent. Considering the matroid $\mathcal{M}(\boldsymbol{A})$ defined over the matrix $\boldsymbol{A}$ of Equation (B.1) the set $\{\boldsymbol{v_2}, \boldsymbol{v_4}\}$ is an independent set, but it is not a basis. On the other hand the set $\{\boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}, \boldsymbol{v_5}\}$ is a dependent set because $\boldsymbol{v_5} = \boldsymbol{v_2} + \boldsymbol{v_3}$. Recalling that a circuit is dependent set, all of whose proper subsets are independent, $\{\boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_5}\}$ is a circuit for matroid $\mathcal{M}(\boldsymbol{A})$. The set $\{\boldsymbol{v_7}\}$ is also a minimal dependent set, thus it is a circuit for the matroid. A single-element circuit is called a loop for the matroid.

Considering the matroid $\mathcal{M}(G)$ defined over the graph G of Figure 53, the set the set $\{e_2, e_4\}$ is an independent set, but it is not a basis. On the other hand the set $\{e_2, e_3, e_4, e_5\}$ is a dependent set because $e_5 = e_2 + e_3$. The set $\{e_2, e_3, e_5\}$ constitutes a circuit for matroid $\mathcal{M}(\boldsymbol{A})$. The set $\{e_7\}$ is also a minimal dependent set, thus it is a circuit for the matroid. Thus $\{e_7\}$ is a loop for the matroid.

Assuming matroids $\mathcal{M}(\boldsymbol{A})$ and $\mathcal{M}(G)$, it can be observed that the sets of bases $\mathcal{B}$ and $\mathcal{T}$ of the two matroids are the same, if the relabelling $\boldsymbol{v_i} \leftrightarrow e_i$ is performed. Thus the following definition can be stated:

Definition 22. *Matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic* ($\mathcal{M}_1 \cong \mathcal{M}_2$) *if there is a one-to-one function ϕ from $E(\mathcal{M}_1)$ onto $E(\mathcal{M}_2)$ that preserves independence; that is, a subset X of $E(\mathcal{M}_1)$ is in $\mathcal{I}(\mathcal{M}_1)$ if and only if $\phi(X)$ is in $\mathcal{I}(\mathcal{M}_2)$, where $\mathcal{I}(M_i)$ is the set of independent sets of the matroid $\mathcal{M}_i$.*

An isomorphism is a structure-preserving correspondence. Thus, two matroids are isomorphic if there is a one-to-one correspondence between their elements that preserves the set of bases.

The size of a matroid is the cardinality of the ground set E, but another characteristic of a matroid is the size of the basis, which is called the rank of the matroid. Thus the rank of matroids $\mathcal{M}(\boldsymbol{A})$ and $\mathcal{M}(G)$ is three. If $S \subset E$ is a set of elements of a matroid, the rank of S is the size of a maximal independent set contained in S. Considering the matroid $\mathcal{M}(\boldsymbol{A})$ and $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_6}, \boldsymbol{v_7}\}$, the rank of S is $r(S) = 2$. The rank of $\{v_7\}$ is zero.

In matroid theory, the dual of a matroid $\mathcal{M}$ is another matroid $\mathcal{M}^*$ with the same ground set E of $\mathcal{M}$ and whose set of bases the dual matroid is precisely the set of all complements of bases of the original matroid. Thus, given a matroid matroid $\mathcal{M}$ on a ground set E, the dual matroid is defined with the same ground set and the set of bases as stated in the following definition:

Definition 23. *Given a matroid $\mathcal{M} = (E, \mathcal{B})$, the dual matroid $\mathcal{M}^* = (E, \mathcal{B}^*)$ can be defined, where $\mathcal{B}^*(\mathcal{M}) = \{(E(\mathcal{M}) \backslash B) \text{ for all } B \in \mathcal{B}(\mathcal{M})\}$.*

Considering the matroid $\mathcal{M}(\boldsymbol{A})$, the dual matroid $\mathcal{M}^*(\boldsymbol{A})$ can be defined as $\mathcal{M}^*(\boldsymbol{A}) = (E, \mathcal{B}^*)$, where:

$$E(\mathcal{M}^*) = E(\mathcal{M}) = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}, \boldsymbol{v_5}, \boldsymbol{v_6}, \boldsymbol{v_7}\} \tag{B.4}$$

and the set of bases of the dual matroid is:

$$\begin{aligned}\mathcal{B}^* = \{&\{\boldsymbol{v_4}, \boldsymbol{v_5}, \boldsymbol{v_6}, \boldsymbol{v_7}\}, \{\boldsymbol{v_3}, \boldsymbol{v_5}, \boldsymbol{v_6}, \boldsymbol{v_7}\}, \{\boldsymbol{v_3}, \boldsymbol{v_4}, \boldsymbol{v_6}, \boldsymbol{v_7}\}, \{\boldsymbol{v_2}, \boldsymbol{v_4}, \boldsymbol{v_6}, \boldsymbol{v_7}\},\\ &\{\boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_6}, \boldsymbol{v_7}\}, \{\boldsymbol{v_1}, \boldsymbol{v_5}, \boldsymbol{v_6}, \boldsymbol{v_7}\}, \{\boldsymbol{v_1}, \boldsymbol{v_4}, \boldsymbol{v_5}, \boldsymbol{v_7}\}, \{\boldsymbol{v_1}, \boldsymbol{v_3}, \boldsymbol{v_6}, \boldsymbol{v_7}\},\\ &\{\boldsymbol{v_1}, \boldsymbol{v_3}, \boldsymbol{v_5}, \boldsymbol{v_7}\}, \{\boldsymbol{v_1}, \boldsymbol{v_3}, \boldsymbol{v_4}, \boldsymbol{v_7}\}, \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_6}, \boldsymbol{v_7}\}, \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_4}, \boldsymbol{v_7}\},\\ &\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_7}\}\}\end{aligned} \tag{B.5}$$

For example the basis $\{\boldsymbol{v_4}, \boldsymbol{v_5}, \boldsymbol{v_6}, \boldsymbol{v_7}\}$ of dual matroid $\mathcal{M}^*(\boldsymbol{A})$ can be obtained as $(E \backslash \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\})$, where $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ is a basis of matroid $\mathcal{M}(\boldsymbol{A})$.

Considering the matroid $\mathcal{M}(G)$, the dual matroid $\mathcal{M}^*(G)$ can be defined as $\mathcal{M}^*(G) = (E, \mathcal{B}^*)$, where the set of bases of the dual matroid is:

$$\begin{aligned}\mathcal{B}^* = \{&\{e_4, e_5, e_6, e_7\}, \{e_3, e_5, e_6, e_7\}, \{e_3, e_4, e_6, e_7\}, \{e_2, e_4, e_6, e_7\},\\ &\{e_2, e_3, e_6, e_7\}, \{e_1, e_5, e_6, e_7\}, \{e_1, e_4, e_5, e_7\}, \{e_1, e_3, e_6, e_7\},\\ &\{e_1, e_3, e_5, e_7\}, \{e_1, e_3, e_4, e_7\}, \{e_1, e_2, e_6, e_7\}, \{e_1, e_2, e_4, e_7\},\\ &\{e_1, e_2, e_3, e_7\}\}\end{aligned} \tag{B.6}$$

For example the basis $\{e_4, e_5, e_6, e_7\}$ of dual matroid $\mathcal{M}^*(G)$ can be obtained as $(E \backslash \{e_1, e_2, e_3\})$, where $\{e_1, e_2, e_3\}$ is a basis of matroid $\mathcal{M}(G)$.

Definition 24. *Bases, circuits, cycles, and independent sets of* $\mathcal{M}^*$ *are called cobases, cocircuits, cocycles, and coindependent sets of* $\mathcal{M}$ *.*

For example $\{\boldsymbol{v_4}, \boldsymbol{v_5}, \boldsymbol{v_6}, \boldsymbol{v_7}\}$ and $\{e_4, e_5, e_6, e_7\}$ are bases of respectively dual matroid $\mathcal{M}^*(A)$ and dual matroid $\mathcal{M}^*(G)$, and are cobases for matroids $\mathcal{M}(A)$ and $\mathcal{M}(G)$.

B.1.1 Greedy algorithm

One important property of matroids is that they are precisely the structures in which the greedy algorithm works successfully. The greedy algorithm tries to solve an optimization problem by always choosing a next step that is locally optimal. In general the greedy algorithm strategy operates in the most short-sighted way possible: it proceeds by taking steps, each of which increases the function by as much as possible. Basically the greedy algorithm at each stage chooses the best option (in terms of a weight function).

In general greedy algorithm is prone to find local optimal solution, failing into find a globally optimal one, because they usually do not operate exhaustively on all the data. For some specific class of problems greedy algorithm can be proven to yield the global optimum. Matroids are exactly one of such classes for which greedy algorithm returns an optimal solution.

In order to perform the greedy algorithm, a positive weight function for the matroid must be defined. A common example of greedy algorithm is the Kruskal algorithm, for finding a minimum-weight spanning tree.

Algorithm 7 Kruskal algorithm for selecting minimum-weight spanning tree

Input: A graph G is given with a positive weight function w on the edges
$B \leftarrow \emptyset$
Sort $E(\mathcal{M})$ into non increasing order by weight w
for each $e \in E$, taken in non increasing order by weight $w(e)$ **do**
    if $B \cup \{e\} \in \mathcal{I}(\mathcal{M})$ **then**
        $B \leftarrow B \cup \{e\}$
return B

In Figure 55 the graph G is presented with weights assigned to each of its edges.

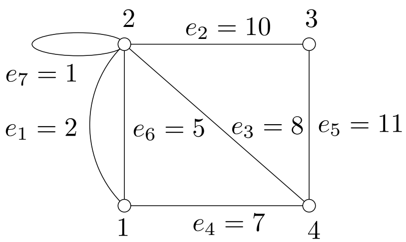


Figure 55 – Connected graph G with weights assigned to each of its edges

The Kruskal Algorithm (7) can be applied to weighted graph G:

1. $B = \emptyset$;
2. The set of edges $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ are sorted in non decreasing order: $\{e_7, e_1, e_6, e_4, e_3, e_2, e_5\}$;
3. The minimum weight element is $\{e_7\}$ but its inclusion would create a circuit, *i.e.* $B = \{e_7\} \notin \mathcal{I}$ is not a minimal set;
4. The next smallest weight edge e_1 is selected. It creates no circuits thus $B = \{e_1\}$;
5. Edge $\{e_6\}$ is considered: it creates circuit with e_1, so it is discarded.
6. Edge $\{e_4\}$ is considered: it creates no circuits with $B = \{e_1\}$, thus it is added to B and $B = \{e_1, e_4\}$;
7. Edge $\{e_3\}$ is considered: it creates circuit with elements of B, so it is discarded;
8. Edge $\{e_2\}$ is considered: it creates no circuits with $B = \{e_1, e_4\}$, thus it is added to B and $B = \{e_1, e_2, e_4\}$;
9. Edge $\{e_5\}$ is considered: it creates circuit with elements of B, so it is discarded;
10. The minimum weight spanning tree $B = \{e_1, e_2, e_4\}$ is returned, *i.e.* the minimum-weight basis for matroid $\mathcal{M}(G)$.

A weight function $w : E \to \mathbb{R}^+$ for a matroid $\mathcal{M}(E, \mathcal{B})$, assigns a strictly positive weight to each element of E. A matroid with an associated weight function is called a weighted matroid. It is important to regard that the steps of Algorithm (7) are not specific to the graph, in fact they all involve avoiding circuits in the matroid. Thus the algorithm constructs a minimum-weight basis for any matroid, graphic or not. Moreover, the following theorems can be enunciated when greedy algorithm is performed on matroids.

Theorem 12. *For any matroid $\mathcal{M}$ and any weight function w, the greedy algorithm return a minimum-weight basis of $\mathcal{M}$.*

A proof can be sketched as follows. Let $\mathcal{M}$ be weighted matroid with weight function w. Suppose that the greedy algorithm generates some basis $B = \{e_1, e_2, \dots, e_n\}$ and that exists some other basis $B' = \{f_1, f_2, \dots, f_n\}$ with smaller total weight. The elements of bases B and B' can be ordered in increasing order. Thus $w(e_1) = w(f_1)$ necessarily. Let k be the smallest integer such that $w(f_k) < w(e_k)$ and consider the two independent sets $I_1 = \{e_1, \dots, e_{k-1}\}$ and $I_1 = \{f_1, \dots, f_k\}$. Recalling the property of independent sets 1, since $|I_1| < |I_2|$ must be some $f_l, l \leq k$ such that $I_1 \cup f_l$ is independent. But this means that $I_1 \cup f_l$ is both independent and has weight smaller than e_k. This is a contradiction because the greedy algorithm would have selected f_l instead of e_k in constructing B,

It important to regard that a matroid can be defined in terms of greedy algorithm. The following theorem can be stated:

Theorem 13. *For any ground set $E = \{1, 2, \dots, n\}$, and a family of subsets $\mathcal{I} \subseteq 2^E$, the greedy algorithm returns the maximum-weight base for any set of weights $w : E \to \mathbb{R}$ if and only if $\mathcal{M}(E, \mathcal{I})$ is a matroid.*

The following theorem (103) states the complexity of the greedy algorithm running on matroid structures:

Theorem 14. *The complexity of the greedy algorithm is:*

$$O(|E|f(r(\mathcal{M})) + |E|log|E|) \tag{B.7}$$

where $r(\mathcal{M})$ is the rank of the matroid $\mathcal{M}$, $|E|$ is the cardinality of the ground set E of the matroid, f represents a function of computational complexity of an independent test, which is the procedure to test whether the obtained set is independent, and it is called independent test oracle.

The independent test oracle complexity, given by f, is $\mathcal{O}(|E|)$ (104).

B.2 FURTHER CONCEPTS IN MATROID THEORY

In this section, some further concepts and equivalent definitions of matroids are presented, which generalise the concept of matroid. In the previous section, a matroid has been defined as an ordered pair $(E, \mathcal{B})$, where E is the ground set and $\mathcal{B}$ is the set of bases of the matroid.

Alternatively, a matroid $\mathcal{M}$ can be written in terms of the ground set E and all independent subsets of E:

Definition 25. *A matroid $\mathcal{M}$ is an ordered pair $(E, \mathcal{I})$ consisting of a finite set E and a collection $\mathcal{I}$ of all independent subsets of E, satisfying the following three axioms 1, 2 and 3.*

1. **Non-emptiness:** The empty set is in $\mathcal{M}$. (Thus, $\mathcal{M}$ is not itself empty.)
2. **Hereditary:** If a set X is an element of $\mathcal{M}$, then every subset $X' \subseteq X$ is also in $\mathcal{M}$.
3. **Augmentation:** If X and Y are two sets in $\mathcal{M}$ where $|X| > |Y|$, then there is an element $x \in X \setminus Y$ such that $Y \cup x$ is in $\mathcal{M}$, where $|X|$ is the cardinality of X, *i.e.* the number of elements of X.

The collection $\mathcal{I}$ is the set of all independent sets of the ground set E, as shown in Figure 56. The set of bases is $\mathcal{B} \subseteq \mathcal{I}$.
In Figures 57 and 58 Axioms 2 and 3 are depicted.

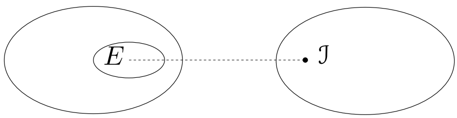


Figure 56 – $\mathcal{I}$ is a collection of E

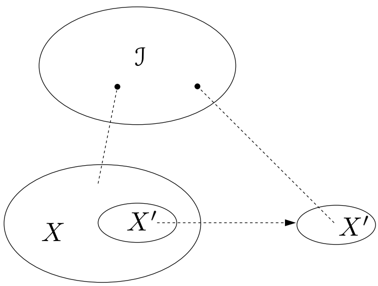


Figure 57 – Hereditary

Different structures satisfy the three axioms 1, 2 and 3, some examples are presented for a better understanding.
Example I Assuming vector independence, the following properties can be stated:

a) $\emptyset \in \mathcal{M}$, the empty set is a independent set (vacuous implication).

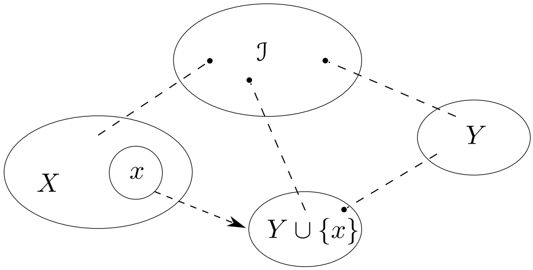


Figure 58 – Augmentation

b) Hereditary: subset of some independent vectors are also independent;

c) Augmentation: a new vector can be added to a smaller independent set to keep independency.

Recalling the column vectors of matrix $\boldsymbol{A}$ in Equation (B.1), it can be verified that:

- Axiom 1 is always satisfied;
- Axiom 2: considering subset $\{\boldsymbol{v_1}, \boldsymbol{v_2}\} \subseteq \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$, it is independent;
- Axiom 3: considering independent sets $I_1 = \{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ and $I_2 = \{\boldsymbol{v_2}, \boldsymbol{v_5}, \boldsymbol{v_6}\}$, they are both independent sets and $|I_1| < |I_2|$. It can be verified that $(I_1 \cup \{\boldsymbol{v_5}\}) \subseteq \mathcal{I}$.

Example II Assuming the set edges of a graph G, the following properties can be stated:

a) $\emptyset$ contains no cycle;

b) Hereditary: subgraph of a cycle-free graph is cycle-free;

c) Augmentation: Adding a new edge to a smaller cycle-free subgraph, a cycle-free subgraph is obtained.

Recalling graph G of Figure 53, it can be verified that:

- Axiom 1 is always satisfied;
- Axiom 2: considering subset $\{e_1, e_2\} \subseteq \{e_1, e_2, e_3\}$, it is independent;

- Axiom 3: considering independent sets $I_1 = \{e_1, e_2\}$ and $I_2 = \{e_2, e_5, e_6\}$, they are both independent sets and $|I_1| < |I_2|$. It can be verified that $(I_1 \cup \{e_5\}) \subseteq \mathcal{I}$.

Example III Assuming points, where collinear or planar points (at least three) are considered dependent, the following properties can be stated:

a) $\emptyset$ is not collinear/ coplanar

b) Hereditary: subset of some non-collinear/coplanar points are still non-collinear/coplanar;

c) Augmentation: a new point can be added to a smaller set of some non-collinear/coplanar points.

Considering the geometry diagram in Figure 59, set of points which belong to the same line, as $\{a, b, c\}$, or are coincident, as $\{d, f\}$, are dependent. On the other hand any other triple not collinear is independent, as for example $\{c, b, f\}$. By definition, any pair of point which are not coincident, are independent. It can be verified that:

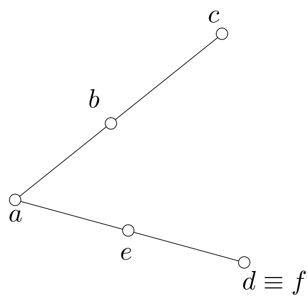


Figure 59 – Geometry diagram

- Axiom 1 is always satisfied;
- Axiom 2: considering the non-collinear set $\{e, b, c\}$, any pair extracted from the subset is still non-collinear or independent.
- Axiom 3: sets $I_1 = \{d, e\}$ and $I_2 = \{e, b, c\}$ are both not collinear sets with $|I_1| < |I_2|$. It can be verified that $\{d, e\} \cup \{b\}$ is a non-collinear set;

Equivalent alternative definitions of matroids are presented. A matroid can be defined in terms of its rank function:

Definition 26. *A matroid consists of a non-empty finite set E and an integer-valued function $r :\to E$ defined on the set of subsets of E, satisfying:*

a) $0 \leq r(\boldsymbol{A}) \leq |A|$, *for each subset A of E;*

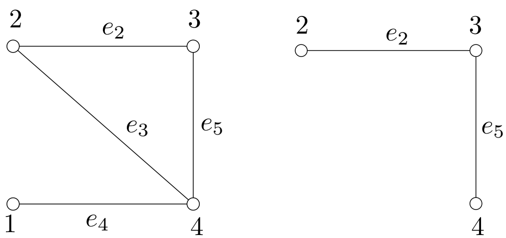
b) *If $A \subseteq B \subseteq E$, then $r(\boldsymbol{A}) \leq r(B)$;*

c) *For any $A, B \subseteq E$, than $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$*

In linear algebra the rank function of a set X of vectors is defined as the rank of X, *i.e.* the maximum number of linearly independent vectors. Recalling the matroid defined over matrix $\boldsymbol{A}$ of equation (B.1), the subsets $C = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\}$ and $D = \{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ The properties of Definition (26) can be verified:

- *Property a)*: $r(C) = 3$ and $|c| = 4$ thus $0 \leq r(C) \leq |C|$;
- *Property b)*: $D \subseteq C \subseteq E$ and $2 = r(D) \leq r(C) \leq r(E) = 3$;
- *Property c)*: $r(C \cup D) + r(C \cap D) \leq r(C) + r(D)$ is $3+2 \leq 3+2$.

Considering the edges of a graph, the rank function can be defined for each subgraph as the maximal number of edges in the subgraph which do not form a cycle. Recalling the graph G of Figure 53, the subgraphs $C\{e_2, e_5, e_3, e_4\}$ and $D = \{e_2, e_5\}$, presented respectively in Figure 60a and 60b considered.



(a) Subgraph C (b) Subgraph D

Figure 60 – Subgraphs C and D of graph G

The subgraph D contains no cycle and it is connected, thus $r(D) = 2$. The subgraph C contains four elements and one cycle. The maximum number of edge which does not contains a cycle is three ($\{e_2, e_5, e_4\}$, $\{e_2, e_3, e_4\}$ or $\{e_5, e_3, e_4\}$) thus $r(C) = 3$. The properties of Definition (26) can be verified:

- *Property a)*: $r(C) = 3$ and $|c| = 4$ thus $0 \leq r(C) \leq |C|$;
- *Property b)*: $D \subseteq C \subseteq E$ and $2 = r(D) \leq r(C) \leq r(E) = 3$;
- *Property c)*: $r(C \cup D) + r(C \cap D) \leq r(C) + r(D)$ is $3+2 \leq 3+2$.

A matroid can be defined in terms of cycles:

Definition 27. *A matroid M consists of a non-empty finite set E , and a collection $\mathcal{C}$ of non-empty subsets of E (called cycles) satisfying the following properties:*

a) *no cycle properly contains another cycle;*

b) *if C_1 and C_2 are two distinct cycles each containing an element e, then there exists a cycle in $C_1 \cup C_2$ that does not contain $\{e\}$.*

Considering a set of column vectors a cycle, or a circuit, is the set of minimally dependent vectors. Recalling the matroid defined over matrix $\boldsymbol{A}$ of equation (B.1) and the matroid defined over graph G in Figure 53, the set of circuits (as the matroids are isomorphic the set of circuits is the same, just relabelling $e_i \leftrightarrow v_i$ can be written as:

$$\begin{aligned}\mathcal{C} = \{&\{\boldsymbol{v_7}\}, \{\boldsymbol{v_1}, \boldsymbol{v_3}, \boldsymbol{v_4}\}, \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_4}, \boldsymbol{v_5}\}, \{\boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_5}\}, \{\boldsymbol{v_2}, \boldsymbol{v_4}, \boldsymbol{v_5}, \boldsymbol{v_6}\},\\ &\{\boldsymbol{v_3}, \boldsymbol{v_4}, \boldsymbol{v_6}\}, \{\boldsymbol{v_1}, \boldsymbol{v_6}\}\}\end{aligned} \tag{B.8}$$

The circuits of the matroid defined over graph G correspond to the cycles of the graph. Property *a)* holds for the set of circuits in Equation (B.8). Considering cycles $C_1 = \{\boldsymbol{v_1}, \boldsymbol{v_3}, \boldsymbol{v_4}\}$ and $C_2 = \{\boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_5}\}$ both contain element $\boldsymbol{v_3}$ and $C_1 \cup C_2 = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_4}, \boldsymbol{v_5}\}$. Thus property *b)* holds.

When a matroid is isomorphic to the cycle matroid of some graph we say it is *graphic*. A matroid that is isomorphic to the vector matroid of some matrix (over some field) is called *representable*. And not every matroid is graphic, nor is every matroid representable.

The uniform matroid, mentioned in Table (13) can be defined as a ground set E of n elements and the set of bases $\mathcal{B}$ formed by all subsets of E with exactly k elements. This matroid is denoted as $\mathcal{U}_{k,n}$.

In this matroid, any set with k elements is a maximal independent set, any set with fewer than k elements is independent, and any set with more than k elements is dependent. Thus the circuits are the sets of size $k+1$.

An example is presented. The ground set is $E = \{a, b, c, d\}$ and $k = 2$. Thus uniform matroid $\mathcal{U}_{2,4}$ can be defined. The bases of this matroid are all sets with two elements. The uniform matroid $\mathcal{U}_{2,4}$ is not graphic, because it is not isomorphic to any graph. In fact it is not possible to construct a graph with four edges such that each collection of three edges form a cycle. On the other hand this matroid is representable because it is isomorphic to the matroid $\mathcal{M}(A)$ defined over matrix $\boldsymbol{A}$:

$$\boldsymbol{A} = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{array} \right]_{3,7} \end{array} \tag{B.9}$$

Examples of different classes of matroids are presented in Table 13 (100).

Table 13 – Classes of matroids

matroid $\mathcal{M}$	ground set E	independent sets $\mathcal{I}(\mathcal{M})$	bases $\mathcal{B}(\mathcal{M})$	circuits $\mathcal{C}(\mathcal{M})$
uniform matroid, $U_{m,n}$ $(0 \leq m \leq n)$	$\{1, 2, \ldots, n\}$	$\{I \subseteq E : \lvert I\rvert \leq m\}$	$\{B \subseteq E : \lvert B\rvert = m\}$	$\{\mathcal{C} \subseteq E : \lvert\mathcal{C}\rvert = m+1\}$
$\mathcal{M}(G)$, cycle matroid of graph G	$E(G)$, edge-set of G	$\{I \subseteq E(G) \mid I$ contains no cycle$\}$	For connected G: edge-sets of spanning trees	edge-sets of cycles
$\mathcal{M}[\boldsymbol{A}]$, vector matroid of matrix $\boldsymbol{A}$ over field F	column labels of $\boldsymbol{A}$	$\{I \subseteq E \mid I$ labels a linearly independent multiset of columns$\}$	labels of maximal independ-ent sets of columns	labels of minimal dependent sets of columns

It important to notice that a matroid $\mathcal{M}$ can be defined over different fields. In this case the ground set of the matroid is the same, but the independent sets are different. Consider the matroid $\mathcal{M}$ with the matrix $\boldsymbol{A}$ analysed, defined over the field $\mathcal{F}_2$.

$$\boldsymbol{A} = \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right]_{3,7} \end{array} \tag{B.10}$$

In this field, the set $\{d, e, f\}$ is no more independent, as any of the vectors of the set can be obtained as the sum of the other two vectors (recall that over field $\mathcal{F}_2$, $1 + 1 = 0$ and $1 + 0 = 1$). In fact $\{d, e, f\}$ is a minimal dependent set on this field, *i.e.* it is a circuit in the matroid. This matroid presents the following characteristics:

- Given any two distinct elements, there is a unique third element that completes a 3-element circuit (That is, any two elements determine a 3-element circuit);
- Any two 3-element circuits will intersect in a single element;
- There is a set of four elements no three of which form a circuit.

The properties described above are precisely the axioms for a finite projective plane. A finite projective plane is an ordered pair, $(\mathcal{P}, \mathcal{L})$, of a finite set $\mathcal{P}$ (points) and a collection $\mathcal{L}$ (lines) of subsets of $\mathcal{P}$ satisfying the following:

- Two distinct points of $\mathcal{P}$ are on exactly one line;
- Any two lines from $\mathcal{L}$ intersect in a unique point;
- There are four points in $\mathcal{P}$, no three of which are collinear.

Elements of the matroid are the points of the geometry, and 3-element circuits of the matroid are lines of the geometry. The example of Equation (B.10) has seven points, and this particular projective plane is called the *Fano plane*, denoted F_7. The Fano plane is shown in Figure 61, with each point labelled by its associated vector over $\mathcal{F}_2$. Viewed as a matroid, any three points on a line (straight or curved) form a circuit.

Considering the set of points:

$$\left\{ d = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, e = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, f = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \tag{B.11}$$

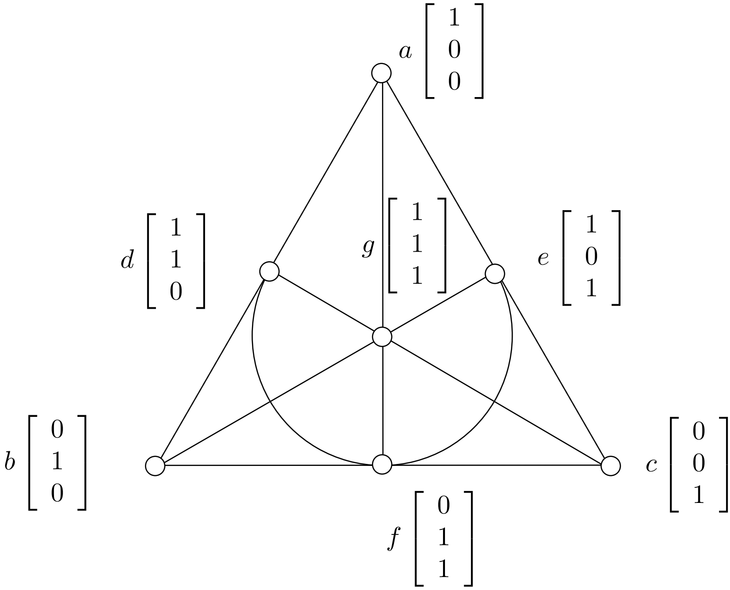


Figure 61 – Fano plane

it can be verified that this set is dependent, as:

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \tag{B.12}$$

over the field $\mathcal{F}_2$.

APPENDIX C – Matrices and complementary results

C.1 SINGLE-LOOP SPATIAL MECHANISM

Consider the mechanism in Figure 62:

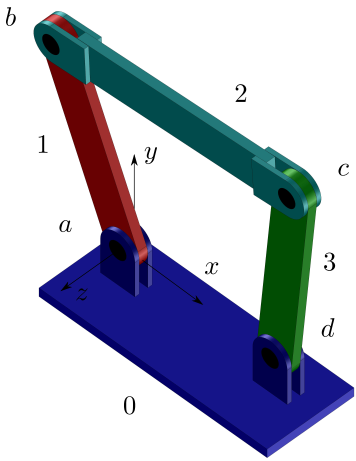


Figure 62 – Four-bar mechanism

with four revolute coupling a, b, c and d, each one *dof* and five *doc*.

C.1.1 Kinematics analysis

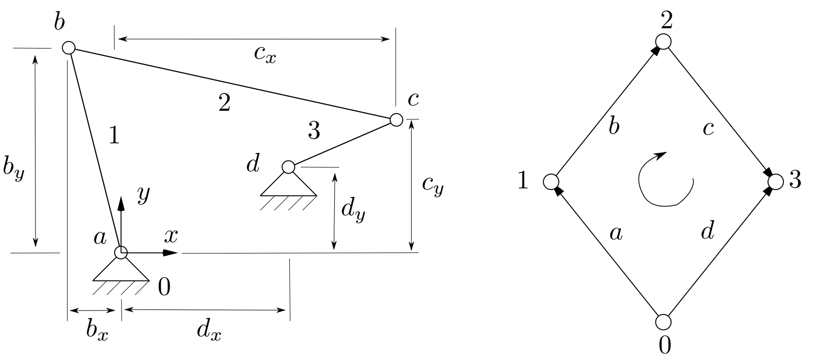
Each one of the joints a, b, c and d allows one single degree of freedom, *i.e.* $f_a = f_b = f_c = f_d = 1$. Thus the motion graph G_M is coincident with he coupling graph G_C, as showed in Figure 63b. The motion screws associated with the joints can be written in the form:

$$\boldsymbol{\$}^{\boldsymbol{m}}_{\boldsymbol{a}} = \begin{bmatrix} 0 \\ 0 \\ \omega_a \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{\$}^{\boldsymbol{m}}_{\boldsymbol{b}} = \begin{bmatrix} 0 \\ 0 \\ \omega_b \\ b_y\omega_b \\ -b_x\omega_b \\ 0 \end{bmatrix} \quad \boldsymbol{\$}^{\boldsymbol{m}}_{\boldsymbol{c}} = \begin{bmatrix} 0 \\ 0 \\ \omega_c \\ c_y\omega_c \\ -c_x\omega_c \\ 0 \end{bmatrix} \quad \boldsymbol{\$}^{\boldsymbol{m}}_{\boldsymbol{d}} = \begin{bmatrix} 0 \\ 0 \\ \omega_d \\ d_y\omega_d \\ -d_x\omega_d \\ 0 \end{bmatrix} \tag{C.1}$$

or as a magnitude multiplied a unit screw in the form:

$$\begin{aligned}
\boldsymbol{\$}^{\boldsymbol{m}}_{\boldsymbol{a}} = \omega_a \hat{\boldsymbol{\$}}^{\boldsymbol{m}}_{\boldsymbol{a}} = \omega_a \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{\$}^{\boldsymbol{m}}_{\boldsymbol{b}} = \omega_b \hat{\boldsymbol{\$}}^{\boldsymbol{m}}_{\boldsymbol{b}} = \omega_b \begin{bmatrix} 0 \\ 0 \\ 1 \\ b_y \\ -b_x \\ 0 \end{bmatrix} \\
\boldsymbol{\$}^{\boldsymbol{m}}_{\boldsymbol{c}} = \omega_c \hat{\boldsymbol{\$}}^{\boldsymbol{m}}_{\boldsymbol{c}} = \omega_c \begin{bmatrix} 0 \\ 0 \\ 1 \\ c_y \\ -c_x \\ 0 \end{bmatrix} \quad \boldsymbol{\$}^{\boldsymbol{m}}_{\boldsymbol{d}} = \omega_d \hat{\boldsymbol{\$}}^{\boldsymbol{m}}_{\boldsymbol{d}} = \omega_d \begin{bmatrix} 0 \\ 0 \\ 1 \\ d_y \\ -d_x \\ 0 \end{bmatrix}
\end{aligned} \tag{C.2}$$

where b_x, b_y, c_x, c_y, d_x and d_y are the joint position coordinates in the coordinate system of Figure 63b.



(a) Structural representation

(b) Coupling graph G_C and motion graph G_M

Figure 63 – Four-bar mechanism structural representation (a), coupling graph G_C (b) coincident with motion graph G_M

Thus the unit motion matrix $\left[\hat{\boldsymbol{M}}_{\boldsymbol{D}}\right]$ can be written as:

$$\hat{\boldsymbol{M}}_{\boldsymbol{D}} = \begin{array}{c} \begin{array}{cccc} a_t & b_t & c_t & d_t \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & b_y & c_y & d_y \\ 0 & -b_x & -c_x & -d_x \\ 0 & 0 & 0 & 0 \end{bmatrix}_{6,4} \end{array} \tag{C.3}$$

The circuit matrix $\boldsymbol{B_M}$ for the mechanism motion graph G_M, presented in Figure 63b is:

$$\boldsymbol{B_M} = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix}_{1,4} \end{array} \tag{C.4}$$

and the unit network motion matrix $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$ can be written as:

$$\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right] = \left[\left[\hat{\boldsymbol{M}}_{\boldsymbol{D}}\right] \cdot \operatorname{diag}\left(\boldsymbol{B_M}\right)\right]$$

$$\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right] = \begin{array}{c} \begin{array}{cccc} a_t & b_t & c_t & d_t \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & b_y & c_y & -d_y \\ 0 & -b_x & -c_x & d_x \\ 0 & 0 & 0 & 0 \end{bmatrix}_{6,4} \end{array} \tag{C.5}$$

The vector of magnitudes is:

$$[\boldsymbol{\psi}] = \begin{bmatrix} \omega_a \\ \omega_b \\ \omega_c \\ \omega_d \end{bmatrix}_{4,1} \tag{C.6}$$

and applying the circuit law Equation (2.10) can be written as:

$$\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]_{6,4}[\boldsymbol{\psi}]_{4,1}=[\mathbf{0}]_{6,1}$$

$$\begin{array}{cccc} a_t & b_t & c_t & d_t \end{array}$$
$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & b_y & c_y & -d_y \\ 0 & -b_x & -c_x & d_x \\ 0 & 0 & 0 & 0 \end{bmatrix}_{6,4} \begin{bmatrix} \omega_a \\ \omega_b \\ \omega_c \\ \omega_d \end{bmatrix}_{4,1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{6,1} \tag{C.7}$$

Davies(17) proposes the following equation for calculating the degree of freedom of a mechanism:

$$F_N = F - \operatorname{rank}(\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]) \tag{C.8}$$

Thus, for the mechanism of Figure 62 the degree of freedom is:

$$F_N = 4 - \operatorname{rank}(\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]) = 4 - 3 = 1 \tag{C.9}$$

Therefore, the unknown magnitudes of motion screws can be expressed in term of a primary variable. Consider that joint d as actuated in the mechanism, thus ω_d can be regarded as a primary variable. Eliminating the zero rows and by means of elementary row operations, the homogeneous linear system of Equation C.7 can be written as:

$$\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]_{6,4}[\boldsymbol{\psi}]_{4,1}=[\mathbf{0}]_{6,1}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & \frac{c_y}{b_y} & -\frac{d_y}{b_y} \\ 0 & 0 & 1 & -\frac{b_x d_x - b_y d_x}{b_x c_y - b_y c_x} \end{array}\right]_{3,4} \begin{bmatrix} \omega_a \\ \omega_b \\ \omega_c \\ -- \\ \omega_d \end{bmatrix}_{4,1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{3,1} \tag{C.10}$$

and isolating the primary variable:

$$\begin{bmatrix} \omega_a \\ \omega_b \\ \omega_c \end{bmatrix}_{3,1} = - \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{c_y}{b_y} \\ 0 & 0 & 1 \end{bmatrix}_{3,3} - \begin{bmatrix} -1 \\ -\frac{d_y}{b_y} \\ -\frac{b_x d_x - b_y d_x}{b_x c_y - b_y c_x} \end{bmatrix}_{3,1} \omega_d \tag{C.11}$$

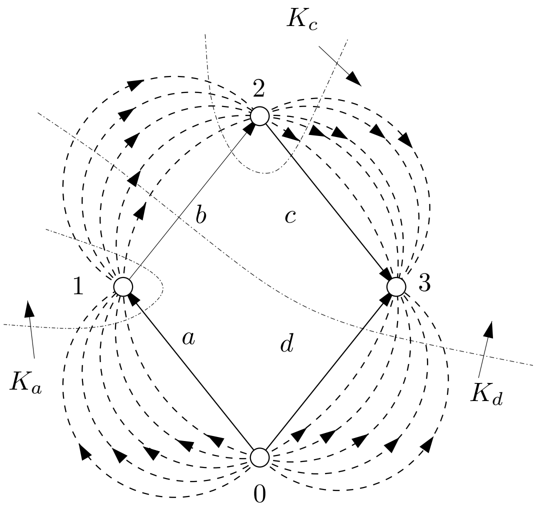


Figure 64 – Action graph G_A of Mechanism in Figure 62.

where all unknown magnitudes are expressed in terms of the primary variable ω_d.

C.1.2 Action analysis

The action graph G_A of the mechanism is presented in Figure 64, where the number of parallel edges joining two links represent the number of *doc* for the coupling.

Thus unit action matrix $\left[\hat{\boldsymbol{A_D}}\right]_{6,20}$ can be written as in Equation (C.12), where lines partitioning matrix $\left[\hat{\boldsymbol{A_D}}\right]_{6,20}$ identify the constraints of couplings a, b, c and d. Based on graph G_A, the number of cutsets is $k = 3$, and the cutset matrix $\boldsymbol{Q_A}$ is presented in Equation (C.13). The cut law for the mechanism can now be written ain Equation(C.14).Equation (C.14) can eventually be written in *rref*, as in Equation (C.15).

$$\left[\hat{\boldsymbol{A}}_{\boldsymbol{D}}\right]_{6,20} = \begin{array}{c} \begin{array}{ccccc|ccccc|ccccc|ccccc} a_R & a_S & a_U & a_V & a_W & b_R & b_S & b_U & b_V & b_W & c_R & c_S & c_U & c_V & c_W & d_R & d_R & d_U & d_V & d_W \end{array} \\ \left[\begin{array}{ccccc|ccccc|ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & -2 & 2 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \tag{C.12}$$

$$[\boldsymbol{Q}_{\boldsymbol{A}}]_{3,20} = \begin{array}{c} \begin{array}{ccccc|ccccc|ccccc|ccccc} a_R & a_S & a_U & a_V & a_W & b_R & b_S & b_U & b_V & b_W & c_R & c_S & c_U & c_V & c_W & d_R & d_R & d_U & d_V & d_W \end{array} \\ \left[\begin{array}{ccccc|ccccc|ccccc|ccccc} 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right] \end{array} \tag{C.13}$$

$$
\begin{array}{c}
\begin{array}{ccccc|ccccc|ccccc|ccccc}
a_R & a_S & a_U & a_V & a_W & b_R & b_S & b_U & b_V & b_W & c_R & c_S & c_U & c_V & c_W & d_R & d_R & d_U & d_V & d_W
\end{array} \\
\left[\begin{array}{ccccc|ccccc|ccccc|ccccc}
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & -0 & 0 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]_{18,20}
\end{array}
\begin{bmatrix} R_a \\ S_a \\ U_a \\ V_a \\ W_a \\ R_b \\ S_b \\ U_b \\ V_b \\ W_b \\ R_c \\ S_c \\ U_c \\ V_c \\ W_c \\ R_d \\ S_d \\ U_d \\ V_d \\ W_d \end{bmatrix}_{C,1}
=
\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{1,18}
\tag{C.14}
$$

$$
\begin{array}{c}
\begin{array}{ccccc|ccccc|ccccc|ccccc}
a_R & a_S & a_U & a_V & a_W & b_R & b_S & b_U & b_V & b_W & c_R & c_S & c_U & c_V & c_W & d_R & d_R & d_U & d_V & d_W
\end{array} \\
\left[\begin{array}{ccccc|ccccc|ccccc|ccccc}
\boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -0 \\
0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & -0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]_{18,20}
\end{array}
\left[\begin{array}{c}
\boxed{R_a} \\ \boxed{S_a} \\ \boxed{U_a} \\ \boxed{V_a} \\ \boxed{W_a} \\ \boxed{R_b} \\ \boxed{S_b} \\ \boxed{U_b} \\ \boxed{V_b} \\ \boxed{W_b} \\ \boxed{R_c} \\ \boxed{S_c} \\ \boxed{U_c} \\ \boxed{V_c} \\ \boxed{W_c} \\ R_d \\ S_d \\ \boxed{U_d} \\ \boxed{V_d} \\ W_d
\end{array}\right]_{20,1}
=
\left[\begin{array}{c}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{array}\right]_{18,1}
\tag{C.15}
$$

C.2 PLANAR MECHANISM

In this section the matrices relative to the planar mechanism shown in Figure 65 are presented.

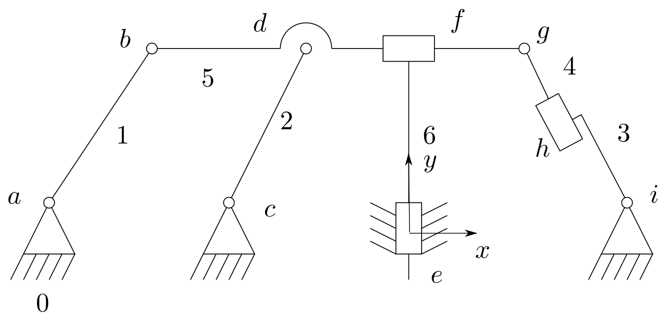


Figure 65 – Planar mechanism with $F_N = 1$ and $C_N = 1$

$$\left[\hat{\boldsymbol{A}}_{\boldsymbol{D}}\right] = \begin{array}{c} \\ T \\ U \\ V \end{array} \begin{array}{c} \begin{array}{cccccccccccccccccc} a_U & a_V & b_U & b_V & c_U & c_V & d_U & d_V & e_R & e_U & f_R & f_V & g_U & g_V & h_R & h_U & i_U & i_V \end{array} \\ \left[\begin{array}{cccccccccccccccccc} 0 & -6 & -2 & -4 & 0 & -4 & -2 & -2 & 1 & 0 & 0 & 1 & -2 & 2 & 1 & 0 & 0 & 4 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right]_{3,18} \end{array} \tag{C.16}$$

$$
\left[\hat{\boldsymbol{A}}_{\boldsymbol{N}}\right] =
\begin{array}{c}
\begin{array}{cccccccccccccccccc}
a_U & a_V & b_U & b_V & c_U & c_V & d_U & d_V & e_R & e_U & f_R & f_V & g_U & g_V & h_R & h_U & i_U & i_V
\end{array} \\
\begin{bmatrix}
0 & 6 & -2 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -6 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 6 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -4 \\
-1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 & 0 & 0 & 0 & -4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1
\end{bmatrix}_{18,118}
\end{array}
\tag{C.17}
$$

$$
\begin{array}{c}
\begin{array}{cccccccccccccccccc}
a_U & a_V & b_U & b_V & c_U & c_V & d_U & d_V & e_T & e_U & f_T & f_V & g_U & g_V & h_T & h_U & i_U & i_V
\end{array} \\
\left[\begin{array}{cccccccccccccccccc}
\boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]_{18,18}
\end{array}
\left[\begin{array}{c}
\boxed{U_a} \\ \boxed{V_a} \\ \boxed{U_b} \\ \boxed{V_b} \\ \boxed{U_c} \\ \boxed{V_c} \\ \boxed{U_d} \\ \boxed{V_d} \\ \boxed{T_e} \\ \boxed{U_e} \\ \boxed{T_f} \\ V_f \\ \boxed{U_g} \\ \boxed{V_g} \\ \boxed{T_h} \\ \boxed{U_h} \\ \boxed{U_i} \\ \boxed{V_i}
\end{array}\right]_{18,1}
=
\left[\begin{array}{c}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{array}\right]_{18,1}
\qquad (C.18)
$$

C.3 SPATIAL MECHANISM 3-PPRR

$$\left[\hat{\boldsymbol{M}}_{\boldsymbol{D}}\right] = \begin{array}{c} \begin{array}{cccccccccccc} b_1 & c_1 & e_1 & d_1 & b_1 & c_1 & e_1 & d_1 & b_1 & c_1 & e_1 & d_1 \end{array} \\ \left[\begin{array}{cccccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & -0.86603 & -0.86603 & 0 & 0 & 0.86603 & 0.86603 \\ 0 & 0 & 1 & 1 & 0 & 0 & -0.5 & -0.5 & 0 & 0 & -0.5 & -0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -2 & -0.5 & 0 & 0.5 & 1 & -0.5 & 0 & 0.5 & 1 \\ 0 & 0 & 0 & 0 & 0.86603 & 0 & -0.86603 & -1.73205 & -0.86603 & 0 & 0.86603 & 1.73205 \\ 0 & 1 & 2 & 1 & 0 & 1 & 2 & 1 & 0 & 1 & 2 & 1 \end{array}\right]_{6,12} \end{array} \tag{C.19}$$

$$\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right] = \begin{array}{c} \begin{array}{cccccccccccc} b_1 & c_1 & e_1 & d_1 & b_1 & c_1 & e_1 & d_1 & b_1 & c_1 & e_1 & d_1 \end{array} \\ \left[\begin{array}{cccccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0.86603 & -0.86603 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0.5 & -0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -2 & 0.5 & 0 & -0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.86603 & 0 & 0.86603 & -1.73205 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & -1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.86603 & -0.86603 & 0 & 0 & 0.86603 & -0.86603 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & -0.5 & 0 & 0 & -0.5 & 0.50000 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & -0.5 & 1 & -0.5 & 0 & 0.5 & -1 \\ 0 & 0 & 0 & 0 & -0.86603 & 0 & 0.86603 & -1.73205 & -0.86603 & 0 & 0.86603 & -1.73205 \\ 0 & 0 & 0 & 0 & 0 & -1 & -2 & 1 & 0 & 1 & 2 & -1 \end{array}\right]_{12,12} \end{array} \tag{C.20}$$

In Table 14

Table 14 – Distinct actuation schemes for *3-PPRR* mechanism

Ac. Set	**Ac. Set**	**Ac. Set**
$\{b_i, c_i, b_j, b_k\}$	$\{c_i, d_i, b_j, b_k\}$	$\{c_i, e_i, b_j, b_k\}$
$\{b_i, c_i, b_j, c_k\}$	$\{c_i, d_i, b_j, c_k\}$	$\{c_i, e_i, b_j, c_k\}$
$\{b_i, c_i, b_j, d_k\}$	$\{c_i, d_i, b_j, d_k\}$	$\{c_i, e_i, b_j, d_k\}$
$\{b_i, c_i, b_j, e_k\}$	$\{c_i, d_i, b_j, e_k\}$	$\{c_i, e_i, b_j, e_k\}$
$\{b_i, c_i, c_j, c_k\}$	$\{c_i, d_i, c_j, c_k\}$	$\{c_i, e_i, c_j, c_k\}$
$\{b_i, c_i, c_j, d_k\}$	$\{c_i, d_i, c_j, d_k\}$	$\{c_i, e_i, c_j, d_k\}$
$\{b_i, c_i, c_j, e_k\}$	$\{c_i, d_i, c_j, e_k\}$	$\{c_i, e_i, c_j, e_k\}$
$\{b_i, c_i, d_j, d_k\}$	$\{c_i, d_i, d_j, d_k\}$	$\{c_i, e_i, d_j, d_k\}$
$\{b_i, c_i, d_j, e_k\}$	$\{c_i, d_i, e_j, e_k\}$	$\{c_i, e_i, d_j, e_k\}$
$\{b_i, c_i, e_j, e_k\}$	$\{c_i, d_i, e_j, e_k\}$	$\{c_i, e_i, e_j, e_k\}$

C.4 TRIPTERON MECHANISM

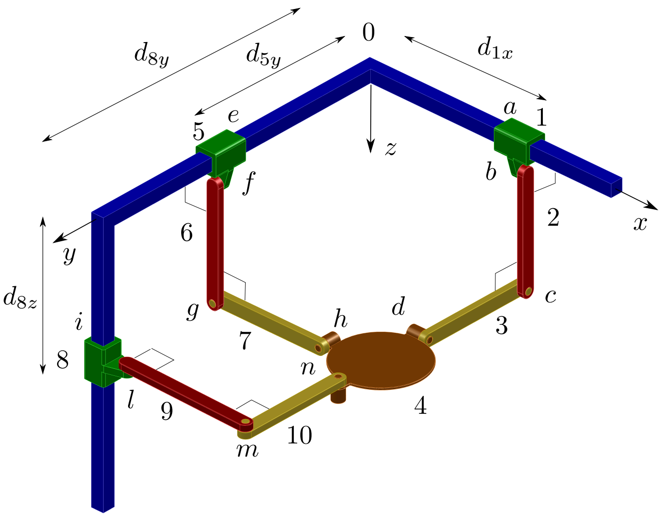


Figure 66 – Mechanism *Tripteron* with mobility $F_N = 3$

Considering the spanning tree of the mechanism presented in Figure 67 (the chords h and d are indicated in dashed line for completeness), the cutset matrix Q for the configuration depicted in Figure 66 is:

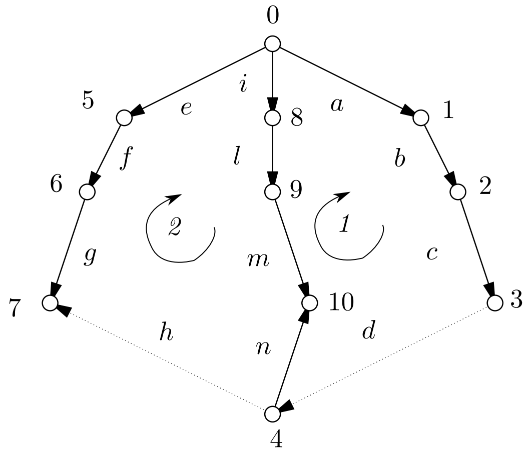


Figure 67 – Graph G_M of *Tripteron* mechanism with mobility $M = 3$

$$[\boldsymbol{Q}]_{10,12} = \begin{array}{c} \\ n \\ m \\ l \\ i \\ g \\ f \\ e \\ c \\ b \\ a \end{array} \begin{array}{c} \begin{array}{cccccccccccc} a & b & c & d & e & f & g & h & i & l & m & n \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \tag{C.21}$$

Matrix $\left[\hat{\boldsymbol{A}}_{\boldsymbol{D}}\right]_{60,12}$ can be written as:

$$\left[\hat{\boldsymbol{A}}_{\boldsymbol{D}}\right]_{60,12} = \left[\begin{array}{ccc} \left[\hat{\boldsymbol{A}}_{\boldsymbol{D}}\right]^{I}_{20,12} & \left[\hat{\boldsymbol{A}}_{\boldsymbol{D}}\right]^{II}_{20,12} & \left[\hat{\boldsymbol{A}}_{\boldsymbol{D}}\right]^{III}_{20,12} \end{array}\right] \tag{C.22}$$

where:

$$\left[\hat{\boldsymbol{A}}_{\boldsymbol{D}}\right]^{I}_{20,12} = \begin{array}{c} \begin{array}{cccccccccccccccccccc} a_R & a_S & a_T & a_V & a_W & b_S & b_T & b_U & b_V & b_W & c_S & c_T & c_U & c_V & c_W & d_S & d_T & d_U & d_V & d_W \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \tag{C.23}$$

$$\left[\hat{\boldsymbol{A}}_{\boldsymbol{D}}\right]^{II}_{20,12} = \begin{array}{c} \begin{array}{cccccccccccccccccccc} e_R & e_S & e_T & e_U & e_W & f_R & f_T & f_U & f_V & f_W & g_R & g_T & g_U & g_V & g_W & h_R & h_T & h_U & h_V & h_W \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \tag{C.24}$$

$$\left[\hat{\boldsymbol{A}}_{\boldsymbol{D}}\right]^{III}_{20,12} = \begin{array}{c} \begin{array}{cccccccccccccccccccc} i_R & i_S & i_T & i_U & i_V & l_S & l_R & l_U & l_V & l_W & m_S & m_R & m_U & m_V & m_W & n_S & n_R & n_U & n_V & n_W \end{array} \\ \left[\begin{array}{cccccccccccccccccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -0 & 0 & -2 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \tag{C.25}$$

C.4.1 Actuation schemes

All feasible actuation schemes for the mechanism *Tripteron*, presented in Figure 37, is herein enumerated by applying Algorithm 2.

Table 15 – Enumeration of Tripteron actuation schemes

Ac. Set	**Ac. Set**	**Ac. Set**	**Ac. Set**
$\{i, m, n\}$	$\{i, a, e\}$	$\{i, c, g\}$	$\{a, b, h\}$
$\{i, a, m\}$	$\{i, n, e\}$	$\{c, n, g\}$	$\{a, m, h\}$
$\{i, a, b\}$	$\{i, m, e\}$	$\{a, c, g\}$	$\{m, n, h\}$
$\{i, b, n\}$	$\{i, e, f\}$	$\{b, c, g\}$	$\{l, m, h\}$
$\{i, b, m\}$	$\{c, e, f\}$	$\{i, b, g\}$	$\{l, n, h\}$
$\{b, m, c\}$	$\{d, e, f\}$	$\{b, m, g\}$	$\{a, l, h\}$
$\{b, c, n\}$	$\{m, d, f\}$	$\{b, n, g\}$	$\{l, f, h\}$
$\{a, b, c\}$	$\{b, d, f\}$	$\{a, b, g\}$	$\{l, g, h\}$
$\{i, a, c\}$	$\{c, d, f\}$	$\{a, m, g\}$	$\{i, l, g\}$
$\{a, m, c\}$	$\{i, c, f\}$	$\{m, n, g\}$	$\{l, m, g\}$
$\{i, c, n\}$	$\{m, c, f\}$	$\{i, m, g\}$	$\{l, n, g\}$
$\{m, n, c\}$	$\{b, c, f\}$	$\{m, g, h\}$	$\{a, l, g\}$
$\{i, m, c\}$	$\{i, b, f\}$	$\{b, g, h\}$	$\{l, c, g\}$
$\{m, d, c\}$	$\{i, m, f\}$	$\{c, g, h\}$	$\{l, d, g\}$
$\{n, c, d\}$	$\{m, f, g\}$	$\{e, g, h\}$	$\{l, f, g\}$
$\{a, c, d\}$	$\{b, f, g\}$	$\{m, f, h\}$	$\{i, l, f\}$
$\{b, m, d\}$	$\{c, f, g\}$	$\{b, f, h\}$	$\{l, c, f\}$
$\{n, b, d\}$	$\{e, f, g\}$	$\{c, f, h\}$	$\{l, d, f\}$
$\{a, b, d\}$	$\{i, e, g\}$	$\{e, f, h\}$	$\{l, m, d\}$
$\{a, m, d\}$	$\{m, e, g\}$	$\{m, e, h\}$	$\{n, l, d\}$
$\{n, m, d\}$	$\{n, e, g\}$	$\{n, e, h\}$	$\{a, l, d\}$
$\{m, d, e\}$	$\{a, e, g\}$	$\{a, e, h\}$	$\{l, m, c\}$
$\{n, d, e\}$	$\{c, e, g\}$	$\{m, c, h\}$	$\{l, c, n\}$
$\{a, d, e\}$	$\{d, e, g\}$	$\{c, n, h\}$	$\{a, l, c\}$
$\{m, e, c\}$	$\{m, d, g\}$	$\{a, c, h\}$	$\{i, a, l\}$
$\{c, n, e\}$	$\{b, d, g\}$	$\{b, m, h\}$	$\{i, l, n\}$
$\{a, c, e\}$	$\{c, d, g\}$	$\{b, n, h\}$	$\{i, l, m\}$

C.5 FIVE-BAR MECHANISM

In this section, Davies's equations are applied to the two-loop planar mechanisms (a) and (b) presented in Figure 68.

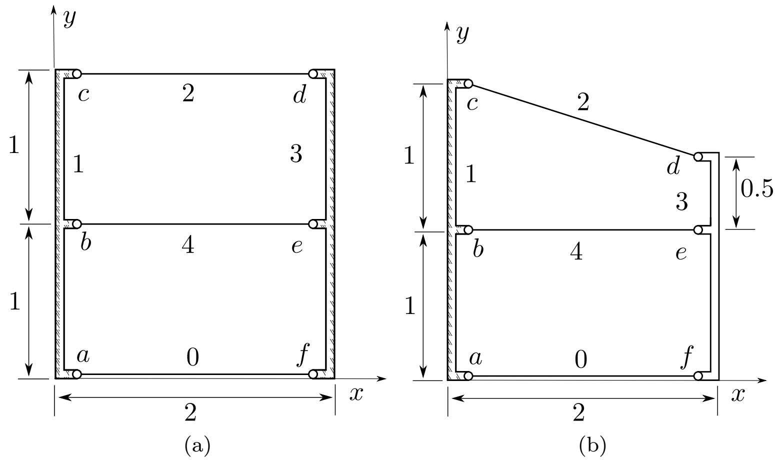


Figure 68 – Two mechanisms with the same number of links and joints, $n = 5$ links and $g = 6$ joints, with $F_N = 1$ and $C_N = 1$ in (a) and $F_N = 0$ and $C_N = 0$ in (b)

For the mechanism in Figure 68a, matrix $\left[\hat{\boldsymbol{M}}_{\boldsymbol{D}}\right]$ can be written as:

$$[\hat{\boldsymbol{M}}_{\boldsymbol{D}}]_{3,6} = \begin{array}{c} \begin{matrix} a & b & c & d & e & f \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 & -2 & -2 \end{bmatrix} \end{array} \tag{C.26}$$

The circuit matrix $\boldsymbol{B_M}$ can be written as:

$$[\boldsymbol{B_M}]_{2,6} = \begin{array}{c} \begin{matrix} a & b & c & d & e & f \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & -1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array} \tag{C.27}$$

Applying Equation (2.12), the network unit motion matrix $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$ is:

$$[\hat{\boldsymbol{M}}_{\boldsymbol{N}}]_{6,6} = \begin{array}{c} \begin{matrix} a & b & c & d & e & f \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & -1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 & -2 & 0 \end{bmatrix} \end{array} \tag{C.28}$$

Applying Davies's equation (3.29), the rank of $[\hat{\boldsymbol{M}}_{\boldsymbol{N}}^T]$ is $\text{rank}([\hat{\boldsymbol{M}}_{\boldsymbol{N}}^T]) = 5$, thus the number of redundant constraints is $C_N = \lambda\nu - r = 3 \cdot 2 - 5 = 1$ for the mechanism in Figure 68a.

For the mechanism in Figure 68b, matrix $\left[\hat{\boldsymbol{M}}_{\boldsymbol{D}}\right]$ can be written as:

$$[\hat{\boldsymbol{M}}_{\boldsymbol{D}}]_{3,6} = \begin{array}{c} \begin{matrix} a & b & c & d & e & f \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1.5 & 1 & 0 \\ 0 & 0 & 0 & -2 & -2 & -2 \end{bmatrix} \end{array} \tag{C.29}$$

The circuit matrix $\boldsymbol{B}_{\boldsymbol{M}}$ is the same of Equation (C.27) and the network unit motion matrix $\left[\hat{\boldsymbol{M}}_{\boldsymbol{N}}\right]$ is:

$$[\hat{\boldsymbol{M}}_{\boldsymbol{N}}]_{6,6} = \begin{array}{c} \begin{matrix} a & b & c & d & e & f \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & -1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 2 & 1.5 & 1 & 0 \\ 0 & 0 & 0 & -2 & -2 & 0 \end{bmatrix} \end{array} \tag{C.30}$$

Applying Davies's equation (3.29), the rank of $[\hat{\boldsymbol{M}}_{\boldsymbol{N}}^T]$ is $\text{rank}([\hat{\boldsymbol{M}}_{\boldsymbol{N}}^T]) = 6$, thus the number of redundant constraints is $C_N = \lambda\nu - r = 3 \cdot 2 - 6 = 0$ for the mechanism in Figure 68b.